THE UPPER BOUND OF THE DIMENSION OF THE SPACE OF ℓ-ADIC POLYLOGARITHMS IN THE FIRST GALOIS COHOMOLOGY

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Abstract. Let K be a number field. Let ℓ be a rational prime and let m be a positive integer greater than 1. We prove a conjecture of Douai and Wojtkowiak giving an explicit upper bound of the dimension of the Q-vector space spanned by ℓ-adic polylogarithms in the first continuous Galois cohomology \( H^1(K, Q(\ell^m)) \). In conjunction with this result, we also give a proof of their descent conjecture for ℓ-adic polylogarithms. The proofs are executed by proving the ℓ-adic analogue of the weak Zagier conjecture. As a bi-product of the proofs of conjectures, we show Goncharov’s “7-terms” relation for the ℓ-adic trigonohrom.

1. Introduction

Let K be a number field. Let ℓ be a rational prime and let m be a positive integer. We put \( Z_\ell(1) := \lim_{\longrightarrow} \mu_{\ell^n} \) and \( Q_\ell(m) := Z_\ell(1)^{\otimes m} \otimes \mathbb{Z}_\ell \), where \( \mu_{\ell^n} \) is the group of \( \ell^n \)-th roots of unity in a fixed algebraic closure \( \overline{K} \) of K. Let \( \zeta_\infty = (\zeta_{\ell^n})_{n \geq 1} \) be a fixed basis of \( Z_\ell(1) \) as a \( \mathbb{Z}_\ell \)-module and let \( \zeta_{\ell^m} \) be the m-fold tensor product of \( \zeta_\infty \). For each \( z \in K \setminus \{0, 1\} \), Z. Wojtkowiak introduced a continuous function called the \( m \)-th ℓ-adic polylogarithm

\[
\ell^m(z, \gamma) : \mathcal{G}_K := \text{Gal}(\overline{K}/K) \to Q_\ell \cong Q_\ell(m)
\]

attached to a \( Q_\ell \)-path \( \gamma : \overline{\mathbb{O}} \to z \) in \( P^1_\mathbb{R} \setminus \{0, 1, \infty\} \) which cuts out some coefficient of a power series attached to \( \gamma \). Here, the last isomorphism in (1.1) is defined by the fixed basis \( \zeta_{\ell^m} \) of \( Q_\ell(m) \). If \( z \) is contained in the group \( \mu(K) \) of roots of unity in \( K \), then there exists a specific path \( \gamma \) such that \( \ell^m(z, \gamma) \) becomes a 1-cocycle whose cohomology class essentially coincides with the Soulé cyclotomic element. However, in contrast to the case \( z \in \mu(K) \), \( \ell^m(z, \gamma) \) is not a 1-cocycle for general \( z \in K \setminus \mu(K) \) and \( m > 1 \) even if we replace \( \gamma \) by another path. Therefore, it is difficult to extract some arithmetic informations from the single ℓ-adic polylogarithm. The idea to overcome such a difficulty, which was initiated by Beilinson–Deligne (cf. [1]), is to take a linear sum of ℓ-adic polylogarithms.

In the paper [6], J.-C. Douai and Wojtkowiak considered \( \mathbb{Q} \)-linear sums of ℓ-adic polylogarithms

\[
\sum_{i=1}^s a_i \ell^m(z_i, \gamma_i) : \mathcal{G}_K \to Q_\ell(m), \quad a_i \in \mathbb{Q}, z_i \in K \setminus \{0, 1\}, \gamma_i : \overline{\mathbb{O}} \to z_i
\]

and studied what condition makes such sums be 1-cocyles. Let \( P^\ell_m(K) \) be the \( \mathbb{Q} \)-vector space consisting of 1-cocyles of \( \mathcal{G}_K \) with coefficients in \( Q_\ell(m) \) of the form (1.2), that is,

\[
P^\ell_m(K) := \left\{ c = \sum_{i} a_i \ell^m(z_i, \gamma_i) \big| a_i \in \mathbb{Q}, z_i \in K \setminus \{0, 1\}, \gamma_i : \overline{\mathbb{O}} \to z_i, c \text{ is a 1-cocycle} \right\}.
\]

The symbol \( [P^\ell_m(K)] \) denotes the image of \( P^\ell_m(K) \) in the first continuous Galois cohomology \( H^1(K, Q_\ell(m)) \). A priori, there is no reason to believe that the dimension of the \( \mathbb{Q} \)-vector space \( [P^\ell_m(K)] \) is finite unless \( H^1(K, Q_\ell(m)) = 0 \) because \( Q_\ell \) is an infinite dimensional \( \mathbb{Q} \)-vector space. However, they conjectured that \( [P^\ell_m(K)] \) is finite dimensional and gave a conjectural explicit upper bound of \( \text{dim}_{\mathbb{Q}} [P^\ell_m(K)] \) as follows:

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CONJECTURE 1.1 ([6, Conjecture 4.1, Conjecture 4.7]). Let \( r_1 \) and \( r_2 \) be the numbers of real and complex places of \( K \), respectively. We put
\[
d_m := \begin{cases} 
    r_1 + r_2 & \text{if } m \text{ is odd,} \\
    r_2 & \text{if } m \text{ is even.}
\end{cases}
\]
If \( m \) is greater than 1, then the following inequality holds:
\[
\dim \mathbb{Q}[P^\ell_m(K)] \leq d_m.
\]
Furthermore, the inequality (1.3) becomes an equality when \( m = 2 \).

The conjectural inequality (1.3) was proved when \( m = 2 \) in loc. cit. Theorem 4.6. The main result of the present paper states that the above conjecture is true:

THEOREM A. Conjecture 1.1 holds for an arbitrary number field \( K \) and an arbitrary rational prime \( \ell \). Moreover, if \( m = 3 \), then the inequality (1.3) becomes an equality.

The proof of inequality (1.3) of Theorem A is executed by proving the \( \ell \)-adic analogue of the weak Zagier conjecture, namely, we will show that all elements of \([P^\ell_m(K)]\) is contained in the image of the \( \ell \)-adic regulator
\[
\text{reg}_{2m-1}^\ell : K_{2m-1}(K) \otimes \mathbb{Z} \mathbb{Q} \to H^1(K, \mathbb{Q}_\ell(m))
\]
in the sense of Soulé (cf. [24]). Furthermore, we will show that the inequality (1.3) becomes an equality if and only if the strong Zagier conjecture holds. See Proposition 6.7.

In the same paper [6], Douai and Wojtkowiak also conjectured that each “motivic” linear sum of \( \ell \)-adic polylogarithms contained in \([P^\ell_m(K)]\) has a descent property:

CONJECTURE 1.2 ([6, Conjecture 0.2]). Let \( L \) be a finite Galois extension of \( K \) with the Galois group \( G = \text{Gal}(L/K) \) and let \( c = \sum a_i \ell_{m}(z_i, \gamma_i) \) be an element of \( P^\ell_m(L) \). If the formal sum \( \sum a_i \{ z_i \} \in \mathbb{Q}[L \setminus \{ 0, 1 \}] \) is \( G \)-invariant, then the cohomology class \([c]\) of \( c \) in \( H^1(L, \mathbb{Q}_\ell(m)) \) is also \( G \)-invariant. Furthermore, \([c]\) is contained in the image of the composite
\[
K_{2m-1}(K) \otimes \mathbb{Z} \mathbb{Q} \xrightarrow{\text{reg}_{2m-1}^\ell} H^1(K, \mathbb{Q}_\ell(m)) \xrightarrow{\sim} H^1(L, \mathbb{Q}_\ell(m))^G.
\]

They proved the first assertion of Conjecture 1.2 assuming certain linearly disjointness condition for \( L \) and \( K(\mu_\infty) := \cup_{n \geq 1} K(\mu_n) \) (cf. [6, Theorem 0.3]) and proved the second assertion when \( m = 2 \). We will show that the above conjecture is also true:

THEOREM B. Conjecture 1.2 holds for an arbitrary number field \( K \), an arbitrary finite Galois extension \( L \) of \( K \), and an arbitrary rational prime \( \ell \).

Theorem B is a direct consequence of the proof of Theorem A. Indeed, we will show that the “correspondence” \( \sum a_i \{ z_i \} \mapsto \sum a_i \ell_{m}(z_i, \gamma_i) \) is contained in \( H^1(L, \mathbb{Q}_\ell(m)) \) is functorial in \( L \). Hence, if the formal sum \( \sum a_i \{ z_i \} \) is \( G \)-invariant, then the corresponding cohomology class \( \sum a_i \ell_{m}(z_i, \gamma_i) \) is also \( G \)-invariant. The second assertion of Conjecture 1.2 follows from the descent property for \( K \)-groups of number fields.

As a bi-product of the proof of Theorem A, we give a criterion for functional equations of \( \ell \)-adic polylogarithms stated below. For each positive integer \( m \), let \( \mathcal{L}_m^\ell : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}(m - 1) \) be the classical \( m \)-th modified polylogarithm which is a generalization of the Bloch–Wigner function (cf. [1, 1.5], [30, p.413 (33)], Subsection 3.4 (3.5)).

THEOREM C. Let \( m \) be a positive integer. Let \( K \) be a number field and let \( \sum a_i \{ z_i \} \) be an element of \( \mathbb{Q}[K \setminus \{ 0, 1 \}] \). If the linear sum \( \sum a_i \mathcal{L}_m^\ell(\sigma(z_i)) \) is equal to 0 for all \( \sigma : K \hookrightarrow \mathbb{C} \), then there exist specific \( \mathbb{Q}_\ell \)-paths \( \gamma_i : 01 \hookrightarrow z_i \) in \( \mathbb{P}^1_{\mathbb{R}} \setminus \{ 0, 1, \infty \} \) such that the restriction of the function \( \sum a_i \ell_{m}(z_i, \gamma_i) \) to \( \mathcal{G}_K(\mu_\infty) \) is identically zero.
Wojtkowiak proved the 5-terms relation of $\ell_1\ell_2$ and the distribution relations of $\ell_i\ell_m$ for all $m$ (cf. [28, Theorem 11.1.14, Corollary 11.2.2]). Since $\mathcal{L}_2^\mathcal{cl}$ and $\mathcal{L}_m^\mathcal{cl}$ satisfy the 5-terms relation and the distribution relations, respectively, Theorem C can be regarded as a generalization of that previous work. As an application of Theorem C, we obtain a new example of functional equation of $\ell$-adic trilogarithm $\ell_3$ on $\mathcal{G}_F$, which is called Goncharov’s “7-terms” functional equation. In Section 7, we prove a stronger result, namely, this “7-terms” relation holds if we replace $K$ by an other general field $F$ whose characteristic does not divide $\ell$.

Finally, we remark on the choice of paths in Theorem C. Let \( \sum_i a_i(z_i) \in \mathbb{Q}[K \setminus \{0,1\}] \) satisfying \( \sum_i a_i\mathcal{L}_i^\mathcal{cl}(\sigma(z_i)) = 0 \) for all $\sigma: K \to \mathbb{C}$ and let $\gamma_i: \hat{\mathbb{O}} \sim z_i$ be a family of $\mathbb{Q}_\ell$-paths. In general, the restriction of the function $\sum_i a_i\ell_i(z_i, \gamma_i)$ to $\mathcal{G}_{\mu(\infty)}$ is not equal to 0 and it depends on the choice of $\gamma_i$. H. Nakamura and Wojtkowiak studied the dependence on the choice of paths under a different condition from us (cf. [18, Theorem 5.7, Corollary 5.8]). Thus it is interesting future subject to study the dependence of a linear sum of $\ell$-adic polylogarithms on the choice of paths under our vanishing condition.

1.1. Plan. The plan of the present paper is as follows. In Section 2, we recall a concept of abstract modified polylogarithm attached to a series of abstract Albanese maps in a mixed Tate category. In the following three sections, we see examples of abstract modified polylogarithms. In Section 3, we define the Hodge modified polylogarithms and give a comparison of that polylogarithms with the classical modified polylogarithms (cf. Proposition 3.10). In Section 4, we define the $\ell$-adic étale modified polylogarithms. We also compare the $\ell$-adic étale modified polylogarithms and Wojtkowiak’s $\ell$-adic polylogarithms. Then, we define the motivic modified polylogarithms in Section 5. In Section 6, we compare these three modified polylogarithms introduced in the previous sections and give proofs of Theorem A and Theorem B. In Section 7, we construct a group homomorphism $r_{5,\infty}$ from $K_5(F)$ to $H^1(F, \mathbb{Q}_\ell(3))$ for an arbitrary field $F$ whose characteristic does not divide $\ell$. Then, we will show that $r_{5,\infty}$ coincides with the Soulé $\ell$-adic higher regulator multiplied by a non-zero rational number if $F$ is a number field. In Appendix A, we give proofs of technical lemmas which are needed to describe classifying spaces of torsors under algebraic groups in a mixed Tate category.

1.2. Notation. For a field $F$, we fix its separable closure $F^{sep}$ and denote by $\mathcal{G}_F$ the absolute Galois group $Gal(F^{sep}/F)$ of $F$. For any topological group $A$ equipped with a continuous action of $\mathcal{G}_F$, we denote by $H^1(F, A)$ the continuous first Galois cohomology. For a set $S$, let $\mathbb{Z}[S]$ be the free abelian group generated by symbols $\{s\}$, $s \in S$.

Let $k$ be a field of characteristic 0 and let $V$ be a finite dimensional $k$-vector space equipped with an algebraic action of the multiplicative group $G_m(k)$. Then, we define $V^{(-2n)}$ to be the subspace of $V$ on which $G_m(k)$ acts via the $n$-th power of the standard character $std := id_{G_m(k)}: G_m(k) \to G_m(k)$. For each abstract group $G$, we denote by $G_k$ the unipotent completion of $G$ over $k$ in the sense of [13, Appendix A]. Let $R$ be a $k$-algebra and let $X$ be a $k$-scheme. We denote by $X_R$ or by $X \otimes_k R$ the base change of $X$ to $Spec(R)$. For a scheme $X$, the symbol $\mathcal{O}(X)$ denotes the ring of regular functions on $X$. We denote by $P^1_{01\infty}$ the scheme $\text{Spec}((\mathbb{Z}[\ell, \frac{1}{(\ell-1)}]) = P^1_{\mathbb{Z}} \setminus \{0, 1, \infty\})$.

We mean a left action by an action unless otherwise noted. Let $G$ be a group and let $A$ be a set equipped with an action of $G$. Then for each $g \in G$ and $a \in A$, we denote by $g^a$ the action of $g$ on $a$. For an object $X$ of a category, the symbol $[X]$ denotes the isomorphism class of $X$.

2. Abstract modified polylogarithms in mixed Tate categories

In this section, we recall abstract modified polylogarithms in a mixed Tate category introduced in [20, Section 2]. This notion was referred as abstract polylogarithm in that previous work.
2.1. Preliminaries on mixed Tate categories. In this and the next subsection, we fix a field \(k\) of characteristic 0 and a mixed Tate category \(\mathcal{M}\) over \(k\) with the invertible object \(k(1)\) (cf. [10, Appendix 8.1]). Any object \(M\) in \(\mathcal{M}\) has a natural weight filtration \(W_\bullet M\) indexed by even integers such that \(\text{Gr}^W_{2n}M := W_{2n}M/W_{2n-2}M\) is isomorphic to a direct sum of \(k(-2n)\). Let \(\omega\) be the canonical fiber functor of \(\mathcal{M}\) defined by

\[
\omega : \mathcal{M} \to \text{GrVec}_k \to \text{Vec}_k; M \mapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}_\mathcal{M}(k(n), \text{Gr}_M^n),
\]

where \(\text{GrVec}_k\) is the category of finite dimensional graded \(k\)-vector spaces. Let \(\pi_1(\mathcal{M}, \omega)\) be the Tannakian fundamental group of \(\mathcal{M}\) with the base point \(\omega\). Since \(\omega\) factors through \(\text{GrVec}_k\), there exists a natural splitting \(\pi_1(\mathcal{M}, \omega) = G_{m,k} \rtimes U(\mathcal{M})\) where \(U(\mathcal{M})\) is a pro-unipotent pro-algebraic group over \(k\). Namely, there exists an inverse system \(\{U_\alpha\}_{\alpha}\) of unipotent algebraic groups over \(k\) such that \(U(\mathcal{M}) = \lim_{\alpha} U_\alpha\). The fundamental Lie algebra \(\text{Lie}(\mathcal{M})\) of \(\mathcal{M}\) is defined to be the inverse limit \(\lim_{\alpha} \text{Lie}(U_\alpha)\) of the inverse system of Lie algebras \(\{\text{Lie}(U_\alpha)\}_\alpha\). The action of \(G_{m,k}\) on \(U(\mathcal{M})\) defines the positive grading on the fundamental \(\text{coLie}\) algebra

\[
\text{coLie}(\mathcal{M}) := \lim_{\alpha} \text{Hom}_k(\text{Lie}(U_\alpha), k) = \bigoplus_{n=1}^\infty \text{coLie}(\mathcal{M})^{(2n)}
\]

of \(\mathcal{M}\) where \(\text{coLie}(\mathcal{M})^{(2n)}\) is the subspace of \(\text{coLie}(\mathcal{M})\) on which \(G_{m,k}\) acts by the \((-n)\)-th power of the standard character \(\text{std} := \text{id}_{G_{m,k}}\). We denote by

\[
d_M : \text{coLie}(\mathcal{M}) \to \bigwedge^2 \text{coLie}(\mathcal{M})
\]

the Lie cobracket of \(\text{coLie}(\mathcal{M})\). The following lemma is well-known.

**Lemma 2.1** (cf. [1, Section 2.1]). We have a canonical isomorphism

\[
\text{coLie}(\mathcal{M})^{(2n)} = \text{Ext}^1_{\mathcal{M}}(k(0), k(n)) \otimes \text{std}^{-n}
\]

as \(G_{m,k}\)-modules. Here, we regard \(\text{Ext}^1_{\mathcal{M}}(k(0), k(n))\) as a \(k\)-vector space equipped with the trivial action of \(G_{m,k}\).

We introduce two conditions for affine group schemes in \(\mathcal{M}\) in the sense of Deligne (cf. [4, Section 5.4]).

**Definition 2.2.** Let \(G = \text{Sp}(A)\) be an affine group scheme in \(\mathcal{M}\). We say that \(G\) satisfies (Pos) (resp. (Triv)) if \(A\) satisfies the following condition:

(\text{Pos}) : \(W_0A = k(0)\) (resp. (\text{Triv}) : \(U(\mathcal{M})\) acts on \(\omega(A)\) trivially).

We say that an affine group scheme \(G = \text{Sp}(A)\) in \(\mathcal{M}\) is an algebraic group in \(\mathcal{M}\) if the \(k\)-algebra \(\omega(A)\) is finitely generated. The canonical fiber functor \(\omega\) induces a functor from the category of algebraic groups in \(\mathcal{M}\) to the category of algebraic groups over \(k\) equipped with an algebraic action of \(\pi_1(\mathcal{M}, \omega)\). We use the same letter \(\omega\) for that functor by abuse of notation. The following proposition is a direct consequence of Corollary A.5.

**Proposition 2.3.** Let \(G\) be an algebraic group in \(\mathcal{M}\) satisfying (Pos). Then, the underlying algebraic group of \(\omega(G)\) is unipotent.

For each algebraic group \(G\) in \(\mathcal{M}\), we define the object \(\text{Lie}(G)\) in \(\mathcal{M}\) by the equation

\[
\omega(\text{Lie}(G)) = \text{Lie}(\omega(G)).
\]

Here, the existence of such an object follows from the Tannakian duality. According to Proposition 2.3, the correspondence \(G \mapsto \text{Lie}(G)\) induces an equivalence between the category of
algebraic groups in \( M \) satisfying (Pos) and the category of nilpotent Lie algebra objects in \( M \) with negative weights.

For the rest of this subsection, we fix an algebraic group \( G \) in \( M \) satisfying (Pos) and (Triv). We denote by \( H^1(M, G) \) the set of isomorphism classes of right \( G \)-torsors in \( M \). Recall that the pointed set \( H^1(M, G) \) is canonically isomorphic to the first rational cohomology \( H^1(\pi_1(M, \omega), \omega(G)) \) (cf. [20, Appendix A6.2]). We recall another description of this pointed set:

**Proposition 2.4** ([3, Proposition 5.2]). Let \( G \) be an algebraic group in \( M \) satisfying (Pos) and (Triv).

1. For each \( G \)-torsor \( X \), there exists a unique rational 1-cocycle \( c_1[X] \) representing the isomorphism class \( [X] \) of \( X \) such that \( c_1[X]|_{G_{m,k}} = 1 \).

2. The correspondence \( [X] \mapsto \log(c_1[X]|_{U(M)}) \) defines a bijection \( \Phi: H^1(M, G) \to \text{Hom}_{k, gp}^{G_{m,k}}(U(M), \omega(G)) \cong \text{Hom}_{k, Lie}^{G_{m,k}}(\text{Lie}(M), \omega(\text{Lie}(G))). \)

Here, \( \log(c_1[X]|_{U(M)}) \) is the Lie homomorphism corresponding to the homomorphism \( c_1[X]|_{U(M)}: U(M) \to \omega(G) \) of group schemes over \( k \).

2.2. **Abstract modified polylogarithms.** For any Lie algebra object \( L \) of \( M \) such that \( \omega(L) \) is nilpotent, we denote by \( \exp(L) \) the associated algebraic group in \( M \).

**Definition 2.5.** Let \( m \) be a non-negative integer. We define the polylogarithmic quotient \( \mathcal{R}_m^M \) in \( M \), which is an algebraic group in \( M \), by

\[
\mathcal{R}_m^M := \exp(p_m^M) := \exp \left( k(1) \times \bigoplus_{n=1}^m k(n) \right).
\]

Here, \( p_m^M := k(1) \times \bigoplus_{n=1}^m k(n) \) is a Lie algebra object in \( M \) such that the abelian Lie algebra \( k(1) \) acts on the abelian Lie algebra \( \bigoplus_{n=1}^m k(n) \) by \( k(1) \otimes k(n) \mapsto n k(n) + 1 \) for \( n < m \) and annihilates \( k(m) \). We understand \( \mathcal{R}_0^M \) as \( \exp(k(1)) \).

It is easily checked that the algebraic group \( \mathcal{R}_m^M \) in \( M \) satisfies two conditions (Pos) and (Triv). By using Proposition 2.4, we define a natural map \( r_m \) which is needed to linearize abstract Albanese maps.

**Lemma 2.6** ([20, Lemma 2.4]). Let \( m \) be a non-negative integer. Then, there exists a natural map of pointed sets

\[
r_m: H^1(M, \mathcal{R}_m^M) \to \text{coLie}(M)^{(2m+2)}
\]

where \( \epsilon \) denotes 0 or 1 when \( m > 0 \) or \( m = 0 \), respectively.

**Proof.** We define the map

\[
r_m: H^1(M, \mathcal{R}_m^M) \xrightarrow{\Phi} \text{Hom}_{k, Lie}^{G_{m,k}}(\text{Lie}(M), \omega(p_m^M)) \to \text{coLie}(M)^{(2m+2)}
\]

by

\[
r_m([X]) := \Phi([X])|_{\text{Lie}(M)^{(-2m-2)}} = \log(c_1[X]|_{\text{Lie}(M)^{(-2m-2)}}).
\]

This map coincides with the map defined in [20, Lemma 2.5].

There exists another description of \( r_m \) by using the weight filtration on \( U(M) \). The action of \( G_{m,k} \) on \( \text{Lie}(M) \) defines a natural filtration \( W_{-2n}\text{Lie}(M) \) by the equality

\[
W_{-2n}\text{Lie}(M) := \text{the closure of } \bigoplus_{i \geq n} \text{Lie}(M)^{(-2i)} \text{ in } \text{Lie}(M).
\]

The graded piece \( \text{gr}_{-2n}^\text{W}\text{Lie}(M) \) is canonically isomorphic to \( \text{Lie}(M)^{(-2n)} \). We denote by \( W_{-2n}U(M) \) the closed sub-pro-algebraic group of \( U(M) \) corresponding to \( W_{-2n}\text{Lie}(M) \), namely,

\[
\text{Lie}(W_{-2n}U(M)) = W_{-2n}\text{Lie}(M).
\]
By definition, \( \log: U(M)(k) \sim \to \text{Lie}(M) \) induces a natural isomorphism of \( k \)-vector spaces 
\[
(\text{gr}^W_{-2n}U(M)) (k) \sim \to \text{gr}^W_{-2n}\text{Lie}(M) \cong \text{Lie}(M)^{(2n)}.
\] 
Since the following lemma had essentially proved in [20, Lemma 2.4] by using a standard weight argument, we skip the proof of it.

**Lemma 2.7.** Let \( c: \pi_1(\mathcal{M}, \omega) \to \omega(\mathcal{P}^m) \) be a rational 1-cocycle. If \( m \) is a positive integer, then the restriction of \( c \) to \( W_{-2n}U(M) \) induces a group homomorphism 
\[
\text{gr}^W_{-2n}(c): (\text{gr}^W_{-2n}U(M)) (k) \to \omega(k(m)) = k.
\]
Furthermore, this homomorphism depends only on the cohomology class \([c]\) in the first rational cohomology \( H^1(\pi_1(\mathcal{M}, \omega), \omega(\mathcal{P}^m)) \).

Let \( X \) be a right torsor under \( \mathcal{P}^m \) and let \( c: \pi_1(\mathcal{M}, \omega) \to \mathcal{P}^m \) be any rational 1-cocycle representing \( X \). Then, by the construction of \( r_m \) and Lemma 2.7, we have
\[
r_m([X]) = \text{gr}^W_{-2n}(c)
\]
under the isomorphism (2.3). Now, we recall the concept of series of abstract Albanese maps. For the rest of this section, we fix a field \( F \) which will be specified as a number field \( K \) in Section 5 and Section 6.

**Definition 2.8** (cf. [20, Definition 2.3, Definition 2.5]).

1. A series of abstract Albanese maps \( \text{Alb} = \{\text{Alb}_n\}_{n \geq 0} \) is an inverse system of maps 
\[
\text{Alb}_n: P^1_{01\infty}(F) = F \setminus \{0, 1\} \to H^1(\mathcal{M}, \mathcal{P}^m)
\]
with respect to \( n \) satisfying the following two conditions:

   **(Hom)** \( \text{Alb}_0 \) extends to a group homomorphism from \( F^\times \) to \( H^1(\mathcal{M}, k(1)) \).

   **(Ref)** We have \( \text{Alb}_1(z) = (\text{Alb}_0(z), \text{Alb}_0(1 - z)) \) in \( H^1(\mathcal{M}, \mathcal{P}^m) = H^1(\mathcal{M}, k(1)) \oplus \mathbb{Z}^2 \) for all \( z \in P^1_{01\infty}(F) \).

2. Let \( m \) be a positive integer. We define the \( m \)-th abstract modified polylogarithm
\[
\mathcal{L}_m(\text{Alb}): \mathbb{Z}[P^1_{01\infty}(F)] \to \text{coLie}(\mathcal{M})^{(2m)}
\]
attached to \( \text{Alb} \) to be the linearization of the composite
\[
r_m \circ \text{Alb}_n: P^1_{01\infty}(F) \to \text{coLie}(\mathcal{M})^{(2m)}.
\]

Abstract modified polylogarithms satisfy the following differential formula.

**Proposition 2.9** ([20, (2.2)], [1, Proposition 2.3]). For each positive integer \( m \), we have the following differential formula:
\[
d_M \mathcal{L}_m(\text{Alb}) = \mathcal{L}_{m-1}(\text{Alb}) \wedge \mathcal{L}_0(\text{Alb}).
\]

The above proposition leads us to define Bloch groups attached to \( \text{Alb} \).

**Definition 2.10** ([20, Definition 2.6]). Let \( m \) be a positive integer.

1. We define \( R_m(F, \text{Alb}) \) the space of functional equations of \( \mathcal{L}_m(\text{Alb}) \) by
\[
R_m(F, \text{Alb}) := \text{Ker} \left( \mathcal{L}_m(\text{Alb}): \mathbb{Z}[P^1_{01\infty}(F)] \to \text{coLie}(\mathcal{M})^{(2m)} \right).
\]
Proposition 2.11, corresponding to $V^N$ dimensional, $N$ linear endomorphisms equivalent to giving a finite dimensional graded $R$ of a mixed Hodge–Tate structure over $R$. We denote by $\text{Alb}(\mathcal{M})$ the $m$-th Bloch group attached to $\mathcal{M}$. Then, $\mathcal{L}_m(\mathcal{M})$ induces a well-defined injective group homomorphism $\mathcal{L}_m(\mathcal{M}) : B_m(F, \mathcal{M}) \hookrightarrow \text{Ext}^1_{\mathcal{M}}(k(0), k(m))$.

Proof. For readers’ convenience, we recall the proof of the second assertion of the proposition briefly. The proof is executed by induction on $m$. If $m = 1$, there is nothing to prove because $\text{coLie}(\mathcal{M})^{(2)} = \text{Ext}^1_{\mathcal{M}}(k(0), k(1)) \otimes \text{std}^{-1}$. Thus, we show the case where $m > 1$. By Proposition 2.9 and Lemma 2.1, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
B_m(F, \mathcal{M}) & \overset{\delta_m}{\longrightarrow} & Z[\mathcal{M}^{1}] / R_m(F, \mathcal{M}) \\
\downarrow & & \downarrow \\
\text{Ext}^1_{\mathcal{M}}(k(0), k(m)) \otimes \text{std}^{-m} & \overset{d_m}{\longrightarrow} & \wedge^2 \text{coLie}(\mathcal{M})^{(2m-2)}.
\end{array}
\]

Hence the dotted arrow in the diagram exists and we have the conclusion.

Remark 2.12. Let $\xi$ be an element of $Z[\mathcal{M}^{1}]$. Then, by the diagram in the proof of Proposition 2.11, $\xi$ is contained in $A_m(F, \mathcal{M})$ if and only if $d_{\mathcal{M}} \mathcal{L}_m(\mathcal{M})(\xi) = 0$.

3. Hodge modified polylogarithms

In the following three sections, we give examples of abstract modified polylogarithms. The first example is the Hodge modified polylogarithm. We apply our general theory to the category $\mathcal{H}$ of mixed Hodge–Tate structures over $R$.

3.1. Preliminaries on mixed Hodge–Tate structures. Recall that an $R$-mixed Hodge structure $H$ is called a mixed Hodge–Tate structure if any non-zero Hodge number is of the form $h^{q,q}$ for some integer $q$. We denote by $\mathcal{H}$ the category of mixed Hodge–Tate structures over $R$. For simplicity of notation, we use the same letter $H$ for the underlying $R$-vector space of a mixed Hodge–Tate structure $H$. According to [1, Section 2.5], giving an object $H$ in $\mathcal{H}$ is equivalent to giving a finite dimensional graded $R$-vector space $V_\bullet = \oplus_{j \in \mathbb{Z}} V_j$ equipped with $R$-linear endomorphisms $N_n(V_\bullet) : V_\bullet \rightarrow V_\bullet$ of degree $n$ for all positive integers $n$. Since $V_\bullet$ is finite dimensional, $N_n(V_\bullet)$ is a zero map for sufficiently large $n$. The mixed Hodge–Tate structure $H$ corresponding to $V_\bullet$ is defined as follows:
The underlying $\mathbb{R}$-vector space of $H$ is defined by
\[ H := \tau(2\pi \sqrt{-1})V_\bullet := \bigoplus_{n \in \mathbb{Z}} V_n \otimes \mathbb{R} \ R(n) \]
and the weight filtration on $H$ is defined by $W_{-2n}H := \oplus_{j \geq n} V_j \otimes \mathbb{R} \ R(j)$.

- The Hodge filtration on $H \otimes \mathbb{C} = V_\bullet \otimes \mathbb{C}$ is defined by
\[ F^i(V_\bullet \otimes \mathbb{C}) := g \left( \bigoplus_{j \leq -i} V_j \otimes \mathbb{C} \right) \]
where $g$ is a unipotent $\mathbb{C}$-linear automorphism on $V_\bullet \otimes \mathbb{C}$ satisfying the equality
\[ \frac{1}{2} \log(gg^{-1}) = \sum_{n=1}^{\infty} N_n(V_\bullet)(2\pi \sqrt{-1})^n \otimes \sqrt{-1}^{-1} \in \mathfrak{gl}(\tau(2\pi \sqrt{-1})V_\bullet) \otimes \mathbb{C}. \]

Here we take the complex conjugate $\bar{g}$ of $g$ with respect to the $\mathbb{R}$-structure $\tau(2\pi \sqrt{-1})V_\bullet$ of $V_\bullet \otimes \mathbb{C}$.

This implies that the fundamental Lie algebra of $\mathcal{H}$ is isomorphic to the nilpotent completion of the free Lie algebra over $\mathbb{R}$ with the set of generators $\{N_n \in \text{Lie}(\mathcal{H})(-2n)\}_{n \geq 1}$. Now, we fix such a set of topological generators $\{N_n\}_{n \geq 1}$ of $\text{Lie}(\mathcal{H})$.

**Example 3.1.** Let $m$ be a positive integer and let $b \in \mathbb{R}$. We consider the graded vector space $V_\bullet = \mathbb{R}e_0 \oplus \mathbb{R}e_m$ equipped with the nilpotent endomorphism
\[ N_m(V_\bullet) : \mathbb{R}e_0 \oplus \mathbb{R}e_m \to \mathbb{R}e_0 \oplus \mathbb{R}e_m; ae_0 \mapsto abe_m, e_m \mapsto 0 \]
of degree $m$. Then, the mixed Hodge–Tate structure $H$ corresponding to $V_\bullet$ is an extension of $\mathbb{R}(0)$ by $\mathbb{R}(m)$ whose isomorphism class is represented by
\[ b(2\pi)^m \sqrt{-1}^{m-1} \in \mathbb{R}(m-1) = \text{Ext}^1_{\mathcal{H}}(\mathbb{R}(0), \mathbb{R}(m)). \]
This implies that the image $f_b^{(2m)}(2m) \in \text{coLie}(\mathcal{H})(2m)$ of $b(2\pi)^m \sqrt{-1}^{m-1}$ under the canonical injection
\[ \mathbb{R}(m-1) = \text{Ext}^1_{\mathcal{H}}(\mathbb{R}(0), \mathbb{R}(m)) = \text{coLie}(\mathcal{H})(2m), d_{\mathcal{H}} = 0 \to \text{coLie}(\mathcal{H}) \]
is characterized by $d_{\mathcal{H}}(f_b^{(2m)}) = 0, f_b^{(2m)}(N_m) = b$, and by $f_b^{(2m)}(N_n) = 0$ for all $n \neq m$.

**3.2. Definition of Hodge modified polylogarithms.** We fix a positive integer $m$ in this subsection.

**Definition 3.2.** We define the $m$-th Hodge polylogarithmic quotient $\mathcal{P}^\text{Hdg}_m$, which is an algebraic group in $\mathcal{H}$, by $\mathcal{P}^\text{Hdg}_m := \mathcal{P}^\text{Hdg}_m := \exp(\mathbb{R}(1) \times \oplus_{n=1}^m \mathbb{R}(n))$.

Let $\pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1))_\mathbb{R}$ be the unipotent completion of the topological fundamental group of $\mathbb{P}_1^{\text{top}}(\mathbb{C}) = \mathbb{C} \setminus \{0, 1\}$ with the base point $\bar{0}1$ over $\mathbb{R}$. Then, by the theory of iterated integrals due to Chen, $\pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1))_\mathbb{R}$ has a natural structure of a group scheme in $\mathcal{H}$. It is well-known that $\mathcal{P}^\text{Hdg}_m$ is a quotient of $\pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1))_\mathbb{R}$ as an affine group scheme in $\mathcal{H}$ (cf. [4, Proposition 16.13]). Let
\[ u^\text{Hdg}_m : \pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1))_\mathbb{R} \to \mathcal{P}^\text{Hdg}_m \]
be the canonical surjective homomorphism of group schemes in $\mathcal{H}$. For each $z \in \mathbb{P}_1^{\text{top}}(\mathbb{C})$, we denote by $\mathcal{P}^\text{Hdg}_m(\bar{0}1, z)$ the pushforward by $u^\text{Hdg}_m$ of the path torsor $\pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1, z))_\mathbb{R}$ under $\pi_1^\text{top}(\mathbb{P}_1^{\text{top}}(\mathbb{C}, \bar{0}1))_\mathbb{R}$. 
DEFINITION 3.3. The \(m\)-th Hodge-Albanese map \(\text{Alb}_{m}^{\text{Hdg}}: \mathbf{P}_{01}^{1}(C) \rightarrow H^{1}(\mathcal{H}, \mathcal{P}_{m}^{\text{Hdg}})\) is defined by

\[
\text{Alb}_{m}^{\text{Hdg}}(z) := [\mathcal{P}_{m}^{\text{Hdg}}(0,\overline{1}, z)] \text{ for all } z \in \mathbf{P}_{01}^{1}(C).
\]

We denote by \(\text{Alb}_{m}^{\text{Hdg}}\) the series of Hodge Albanese maps \(\{\text{Alb}_{m}^{\text{Hdg}}\}_{m=1}^{\infty}\).

It is well-known that \(\mathcal{P}_{m}^{\text{Hdg}}(0,\overline{1}; z)\) is a direct sum of the Hodge realizations of Kummer torsors \(K(z)\) and \(K(1 - z)\) (cf. [4, Proposition 14.2, Proposition 16.26]). Since the Hodge realization of \(K(z)\) is represented by \(\log|z| \in \mathbb{R} = \text{Ext}^{1}_{\mathcal{H}}(\mathcal{R}(0), \mathcal{R}(1))\), the series \(\text{Alb}_{m}^{\text{Hdg}}\) of Hodge Albanese maps is a series of abstract Albanese maps (cf. Definition 2.8 (1)).

DEFINITION 3.4. We define the \(m\)-th Hodge modified polylogarithm

\[
\mathcal{L}_{m}^{\text{Hdg}}: \mathbb{Z}[\mathbf{P}_{01}^{1}(C)] \rightarrow \text{coLie}(\mathcal{H})^{(2m)}
\]
to be \(\mathcal{L}_{m}(\text{Alb}_{m}^{\text{Hdg}})\).

We define \(R_{m}^{\text{Hdg}}(C), A_{m}^{\text{Hdg}}(C), \text{ and } B_{m}^{\text{Hdg}}(C)\) by

\[
R_{m}^{\text{Hdg}}(C) := R_{m}^{\text{Hdg}}(C, \text{Alb}_{m}^{\text{Hdg}}), \quad A_{m}^{\text{Hdg}}(C) := A_{m}(C, \text{Alb}_{m}^{\text{Hdg}}), \quad B_{m}^{\text{Hdg}}(C) := B_{m}(C, \text{Alb}_{m}^{\text{Hdg}}).
\]

Similarly, for each number field \(K\), we define \(R_{m}^{\text{Hdg}}(K)\) and \(A_{m}^{\text{Hdg}}(K)\) to be the inverse images of \(\oplus_{\sigma: K \rightarrow C} \mathcal{R}_{m}^{\text{Hdg}}(C)\) and \(\oplus_{\sigma: K \rightarrow C} \mathcal{A}_{m}^{\text{Hdg}}(C)\) under the inclusion

\[
\mathbb{Z}[\mathbf{P}_{01}^{1}(K)] \xrightarrow{\times \sigma} \bigoplus_{\sigma: K \rightarrow C} \mathbb{Z}[\mathbf{P}_{01}^{1}(C)],
\]

respectively. Then, we define \(B_{m}^{\text{Hdg}}(K) := A_{m}^{\text{Hdg}}(K)/R_{m}^{\text{Hdg}}(K)\). By Proposition 2.11, the Hodge modified polylogarithm \(\mathcal{L}_{m}^{\text{Hdg}}\) induces a well-defined injective group homomorphism

\[
\mathcal{L}_{m}^{\text{Hdg}}: B_{m}^{\text{Hdg}}(C) \hookrightarrow \text{Ext}^{1}_{\mathcal{H}}(\mathcal{R}(0), \mathcal{R}(m)) = \mathcal{R}(m - 1).
\]

3.3. Classifying spaces of torsors in \(\mathcal{H}\). To calculate \(\mathcal{L}_{m}^{\text{Hdg}}\), we study explicit descriptions of classifying spaces of torsors in \(\mathcal{H}\). We fix an algebraic group \(G = \text{Sp}(A)\) in \(\mathcal{H}\) satisfying the condition (Pos) (cf. Definition 2.2). By definition, \(A\) is a finitely generated Hopf algebra object in \(\text{Ind}(\mathcal{H})\) satisfying the condition \(W_{0}A = \mathbb{R}\). In particular, all the Hodge weights of \(A\) are non-negative and \(F^{1}(A \otimes_{\mathbb{R}} C)\) is a Hopf ideal of \(A \otimes_{\mathbb{R}} C\).

LEMMA 3.5. Let \(g\) be a \(C\)-valued point of the underlying \(\mathbb{R}\)-group scheme \(G\). Let \(g^{*}\) be the \(C\)-algebra homomorphism of \(A \otimes_{\mathbb{R}} C\) induced by the left multiplication of \(g\) on \(G\). Then, \(g^{*}\) on \(A \otimes_{\mathbb{R}} C\) preserves the filtration \(W_{n}A \otimes_{\mathbb{R}} C\). Moreover, the induced isomorphism on \(gr_{2n}^{W}A \otimes_{\mathbb{R}} C\) is the identity map for each \(n\).

Proof. This proposition is a direct consequence of Corollary A.2 and Lemma A.4. \(\Box\)

Let \(X = \text{Sp}(B)\) be a right \(G\)-torsor in \(\mathcal{H}\). We say that \(x \in X(C)\) is a Hodge trivialization of \(X\) if the morphism of schemes

\[
G_{C} \rightarrow X_{C}; g \mapsto xg
\]
preserves Hodge filtrations on the rings of regular functions of both-hand sides. By the exactly same argument in [22, Lemma 3.3], all Hodge weights of \(B = O(X)\) are also non-negative. Hence all \(F^{n}(B \otimes_{\mathbb{R}} C)\) are ideals of \(B \otimes_{\mathbb{R}} C\). Put \(F^{0}(X_{C}) := \text{Spec}(B \otimes_{\mathbb{R}} C/F^{1}(B \otimes_{\mathbb{R}} C))\) and put \(F^{0}(G_{C}) := \text{Spec}(A \otimes_{\mathbb{R}} C/F^{1}(A \otimes_{\mathbb{R}} C))\). Remark that \(F^{0}(X_{C})(C)\) coincides with the set of Hodge trivializations of \(X\). Furthermore, \(F^{0}(X_{C})\) has a natural structure of a right \(F^{0}(G_{C})\)-torsor in the usual sense.

LEMMA 3.6. Let \(X\) be a right \(G\)-torsor in \(\mathcal{H}\). Then, there exists a unique Hodge trivialization of \(X\).
Proof. Since $F^0(G_C)$ is the trivial group scheme by the condition (Pos), we have $F^0(X_C) \cong \text{Spec}(C)$. Therefore, the set of Hodge trivializations $F^0(X_C)(C)$ is singleton. \hfill \square

**Proposition 3.7.** There exists a canonical isomorphism of pointed sets

$$\Psi : G(C)/G(R) \sim H^1(H, G).$$

**Proof.** Let us take $g \in G(C)$. Then, we define the right $G$-torsor $G_g$ as follows:

- The underlying affine $R$-scheme of $G_g$ is defined to be $G$ equipped with the right action of $G$ by right translations.
- The weight filtration on $O(G_g)$ is the same as that of $O(G)$.
- The Hodge filtration on $O(G_g) \otimes_R C$ is defined by

$$F^i(O(G_g) \otimes_R C) = g^i(F^i(O(G) \otimes_R C)).$$

According to Lemma 3.5, $g^i$ is the identity map on each graded piece $gr^{W_i}_O(G) \otimes_R C$. Hence, $O(G_g)$ is an algebra object in $\text{Ind}(\mathcal{H})$ and $G_g$ is actually a right $G$-torsor in $\mathcal{H}$. We put $\Psi(g) := [G_g]$. One can check that $G_g \cong G_g'$ if and only if $gG(R) = g'G(R)$. Hence, $\Psi$ induces an injective map $G(C)/G(R) \hookrightarrow H^1(\mathcal{H}, G)$.

We show the surjectivity of $\Psi$. Let $X$ be a $G$-torsor in $\mathcal{H}$. According to Lemma 3.6, there exists a unique Hodge trivialization $p_H \in X(C)$ of $X$. We take $p_w \in X(R)$ an $R$-valued point of $X$. Then, $p_w$ trivializes the weight filtration of $X$, that is, the $G$-equivariant morphism

$$f_{p_w} : G \rightarrow X; g \mapsto p_w g$$

preserves their weight filtrations on the rings of regular functions of both-hand sides by Corollary A.3. Let $g$ be an element of $G(C)$ satisfying $p_w = p_H g$. Then $f_{p_w}$ defines an isomorphism of $G$-torsors between $G_g$ and $X$. Hence we have $\Psi(gG(R)) = [X]$ and this completes the proof of the surjectivity of $\Psi$. \hfill \square

We denote by

$$c^* : G(C) \sim G(C)$$

the group automorphism induced by the complex conjugate $c$ on $C$. Then, the natural map

$$G(C)/G(R) \rightarrow G(C)c^{*=-1}, gG(R) \mapsto gc^*(g^{-1})$$

is bijective. By composing the logarithmic map from $G(C)$ to the Lie algebra of $G_C$, we obtain the canonical isomorphism of pointed sets

$$\Psi' : H^1(H, G) \sim (\text{Lie}(G) \otimes_R C)^{c^{*=-1}} = \text{Lie}(G) \otimes_R R \sqrt{-1}.$$

Remark that $c^*$ acts on $\text{Lie}(G) \otimes_R C$ by id $\otimes c$.

Now, we assume that $G$ satisfies (Triv). By composing $\Phi^{-1}$ in Proposition 2.4 and $\Psi'$ in (3.2), we obtain an isomorphism

$$\Psi' \circ \Phi^{-1} : \text{Hom}_{\text{Lie}}^G(\text{Lie}(H), \omega(\text{Lie}(G))) \sim \text{Lie}(G) \otimes_R R \sqrt{-1}$$

of pointed sets.

**Proposition 3.8.** Let $G$ be an algebraic group in $\mathcal{H}$ satisfying (Pos) and (Triv). Then, the composite of canonical isomorphisms

$$\Psi' \circ \Phi^{-1} : \text{Hom}_{\text{Lie}}^G(\text{Lie}(H), \omega(\text{Lie}(G))) \sim H^1(H, \text{Lie}(G)) \sim \text{Lie}(G) \otimes_R R \sqrt{-1}$$

sends $f \in \text{Hom}_{\text{Lie}}^G(\text{Lie}(H), \omega(\text{Lie}(G)))$ to

$$2 \sum_n f(N_n)(2\sqrt{-1})^n \otimes \sqrt{-1}^{-1} \in \text{Lie}(G) \otimes_R R \sqrt{-1}.$$
Proof. According to Lemma A.6, the Lie homomorphism
\[ \iota_N: \text{Lie}(G) \otimes_R C \to \text{End}_C(W_{2N}A \otimes_R C); l \mapsto \log \left( \exp(l)^2_{|W_{2N}(A \otimes_R C)} \right) \]
is injective for sufficiently large \( N \). Let us denote by \( \phi_N \) the composite of \( \Psi' \circ \Phi^{-1} \) and \( \iota_N \). Then, to prove this proposition, it is sufficient to show the equality
\[
\phi_N(f) = 2 \sum_n \log \left( \exp(f(N_n))^2_{|W_{2N}A} \right)(2\pi \sqrt{-1})^n \otimes \sqrt{-1}^{-1}
\]in \( \text{End}_C(W_{2N}A \otimes_R C) \) for sufficiently large \( N \).

Let \( f \) be an element of \( \text{Hom}^G_{\text{Lie}}(\text{Lie}(H), \omega(\text{Lie}(G))) \) and let
\[
a_f: \pi_1(H, \omega) \to \text{GL}(\omega(A)); \sigma \mapsto \exp(f(\sigma))^2
\]
be the action of \( \pi_1(H, \omega) \) on \( \omega(A) \) defined by \( f \). Here, \( \exp(f) : \pi_1(H, \omega) \to \omega(G) \) is the group homomorphism corresponding to \( f \). According to Lemma 3.5, \( a_f \) preserves \( \omega(W_{2N}A) \). We denote by \( W_{2N}A_f \) the mixed Hodge–Tate structure on \( \tau(2\pi \sqrt{-1})\omega(W_{2N}A) = W_{2N}A \) defined by the action \( a_f \) (see (3.1)). Then, by definition, \( A_f := \lim_{\rightarrow \to} W_{2N}A_f \) has a natural ring structure and \( \text{Sp}(A_f) \) is a \( G \)-torsor in \( H \) representing \( \Phi^{-1}(f) \in H^1(H, G) \). Let \( h \) be an element of \( \text{Aut}_C(W_{2N}A \otimes_R C) \) such that \( F^i(W_{2N}A_f \otimes_R C) = h(F^i(W_{2N}A \otimes_R C)) \). Then, by (3.1), we have
\[
\frac{1}{2} \log(hh^{-1}) = \sum_n \log(\exp(f(N_n)))^2_{|W_{2N}A}(2\pi \sqrt{-1})^n \otimes \sqrt{-1}^{-1}.
\]
We remark that the left-hand side of the equation (3.4) coincides with \( \frac{1}{2} \phi_N(f) \). Indeed, we may take \( h \) as \( g^f_{|W_{2N}A \otimes_R C} \) where \( g \in G(C) \) is a representative of \( \Psi^{-1}(f) \in G(C)/G(R) \). Therefore, the equality (3.3) holds for all positive integer \( N \) and this completes the proof of the proposition. \( \square \)

3.4. Calculation of \( \mathcal{L}_m^{\text{Hdg}} \). Now we concentrate our attention to \( \mathcal{L}_m^{\text{Hdg}} \).

Corollary 3.9. We define the evaluation map \( \text{ev}_m: \text{coLie}(H)^{(2m)} \to R(m-1) \) by \( f \mapsto f(N_m)(2\pi)^m \sqrt{-1}^{-m-1} \). Then, the following assertions hold:

(1) The evaluation map \( \text{ev}_m \) is the left inverse of the canonical inclusion \( R(m-1) \cong \text{Ext}^1_H(R(0), R(m)) \hookrightarrow \text{coLie}(H)^{(2m)} \).

(2) The diagram
\[
\begin{array}{c}
H^1(H, \mathcal{L}_m^{\text{Hdg}}) \\
\mathcal{L}_m^{\text{Hdg}} \\
\text{coLie}(H)^{(2m)} \\
\end{array}
\begin{array}{c}
\rightarrow \mathcal{L}_m^{\text{Hdg}} \otimes_R R\sqrt{-1} \\
\downarrow \text{pr}_m \\
2\text{ev}_m \\
\rightarrow R(m-1)
\end{array}
\]
commutes, where \( \text{pr}_m \) is the projection to the last component
\[
\mathcal{L}_m^{\text{Hdg}} \otimes_R R\sqrt{-1} = R(0) \times \prod_{n=1}^{m} R(n-1) \rightarrow R(m-1).
\]

Proof. Let \( b \) be a real number. Recall that the element \( f^{(2m)}_b \in \text{coLie}(H)^{(2m)} \) denotes the image of \( b(2\pi)^m \sqrt{-1}^{-m-1} \) under \( R(m-1) \hookrightarrow \text{coLie}(H)^{(2m)} \). Then, by the calculation in Example 3.1, we have equalities
\[
\text{ev}_m(f^{(2m)}_b) = f^{(2m)}_b(N_m)(2\pi)^m \sqrt{-1}^{-m-1} = b(2\pi)^m \sqrt{-1}^{-m-1}.
\]
Thus the assertion (1) of the corollary holds.
The assertion (2) is easily checked by the definition of \( \text{ev}_m \) and Proposition 3.8. \( \square \)

Let us recall the classical modified polylogarithm \( \mathbb{L}_{m}^{\text{cl}}: \mathbf{P}_{01\infty}^{1}(\mathbb{C}) \rightarrow \mathbb{R}(m-1) \), which is a real analytic function defined by

\[
(3.5) \quad \mathbb{L}_{m}^{\text{cl}}(z) := \begin{cases} \\
\sqrt{-1} \text{Im} \left( \sum_{k=0}^{m-1} \frac{B_k}{k!} \log^{k}(z \bar{z}) \text{Li}_{m-k}(z) \right) & \text{if } m \text{ is even}, \\
\text{Re} \left( \sum_{k=0}^{m-1} \frac{B_k}{k!} \log^{k}(z \bar{z}) \text{Li}_{m-k}(z) \right) & \text{if } m \text{ is odd}.
\end{cases}
\]

(cf. [1, 1.5], [30, p.413 (33)]). Here, \( B_m \) is the \( m \)-th Bernoulli number defined by

\[
\sum_{m=0}^{\infty} \frac{B_m}{m!} t^m := \frac{t}{e^t-1}.
\]

**Proposition 3.10.** Let \( m \) be a positive integer. Then, for each \( \sum a_i \{ z_i \} \in A_{m}^{\text{Hdg}}(\mathbb{C}) \), the equality

\[
\sum_i a_i \mathbb{L}_{m}^{\text{Hdg}}(z_i) = - \sum_i a_i \mathbb{L}_{m}^{\text{cl}}(z_i)
\]

holds. In other words, the \( m \)-th Hodge modified polylogarithm \( \mathbb{L}_{m}^{\text{Hdg}}: B_{m}^{\text{Hdg}}(\mathbb{C}) \rightarrow \mathbb{R}(m-1) \) coincides with the classical modified polylogarithm multiplied by \(-1\).

For the proof of Proposition 3.10, we recall the calculation of Beilinson and Deligne in [1] computing the composite of \( \Psi^m: H^1(\mathcal{H}, \mathbb{H}_{m}^{\text{Hdg}}) \rightarrow \mathbf{p}_{m}^{\text{Hdg}} \otimes_{\mathbb{R}} \mathbb{R} \sqrt{-1} \) and the \( m \)-th Hodge Albanese map.

**Lemma 3.11 ([1, Section 1.5]).** For any \( z \in \mathbf{P}_{01\infty}^{1}(\mathbb{C}) \), the equality

\[
(3.6) \quad \text{pr}_m \circ \Psi^m \circ \text{Alb}_{m}^{\text{Hdg}}(z) = -2 \mathbb{L}_{m}^{\text{cl}}(z)
\]

holds in \( \mathbb{R}(m-1) \).

**Proof.** Let \( z \) be an element of \( \mathbf{P}_{01\infty}^{1}(\mathbb{C}) \). According to [7, (2.20)], the image of \( z \) under

\[
\mathbf{P}_{01\infty}^{1}(\mathbb{C}) \rightarrow H^1(\mathcal{H}, \mathbb{H}_{m}^{\text{Hdg}}) \cong (\mathbb{H}_{m}^{\text{Hdg}}(\mathbb{C}))^{c} = -1 \subset \mathbb{H}_{m}^{\text{Hdg}}(\mathbb{C}) = C(1) \times \prod_{n=1}^{m} C(n)
\]

is calculated as

\[
(3.7) \quad (\log(z \bar{z}); -\text{Li}_1^{-}(z), \ldots, -\text{Li}_m^{-}(z)).
\]

Here, \( \text{Li}_m^{-}: \mathbf{P}^1(\mathbb{C}) \rightarrow \mathbb{C} \) is a single-valued and real analytic polylogarithm (see [7, Theorem 2.27]). By the Baker–Campbell–Hausdorff formula, we have the equality

\[
(3.8) \quad \Psi^m \circ \text{Alb}_{m}^{\text{Hdg}}(z) = \left( \log(z \bar{z}); -\text{Li}_1^{-}(z), \ldots, -\sum_{k=1}^{m} \frac{B_k}{k!} \log^{k}(z \bar{z}) \text{Li}_{m-k}^{-}(z) \right).
\]

Then the last component of the right-hand side of (3.8) coincides with the right-hand side of the equation (3.6) by [7, (2.21)]. \( \square \)

**Proof of Proposition 3.10.** Let \( z \) be an element of \( \mathbf{P}_{01\infty}^{1}(\mathbb{C}) \). Then, by Lemma 3.11 and Corollary 3.9 (2), we have the equalities:

\[
2\text{ev}_m \circ \mathbb{L}_{m}^{\text{Hdg}}(z) = 2\text{pr}_m \circ \Psi^m \circ \text{Alb}_{m}^{\text{Hdg}}(z) = -2 \mathbb{L}_{m}^{\text{cl}}(z).
\]

Now we take an element \( \xi = \sum a_i \{ z_i \} \) of \( A_{m}^{\text{Hdg}}(\mathbb{C}) \). According to Corollary 3.9 (1), \( \mathbb{L}_{m}^{\text{Hdg}}(\xi) \) coincides with \( \text{ev}_m(\mathbb{L}_{m}^{\text{Hdg}}(\xi)) \) in \( \mathbb{R}(m-1) \subset \text{coLie}(\mathcal{H})^{(2m)} \). Thus, we have

\[
\mathbb{L}_{m}^{\text{Hdg}}(\xi) = \text{ev}_m(\mathbb{L}_{m}^{\text{Hdg}}(\xi)) = \sum a_i \text{ev}_m(\mathbb{L}_{m}^{\text{Hdg}}(z_i)) = - \sum a_i \mathbb{L}_{m}^{\text{cl}}(z_i).
\]

This completes the proof of Proposition 3.10. \( \square \)
4. $\ell$-adic étale modified polylogarithms

In this section, we fix a rational prime $\ell$ and a field $F$ of characteristic $p \geq 0$ satisfying the following condition:

**Example 4.1.** If $F$ is finitely generated over the prime field $k_p$ of characteristic $p \nmid \ell$, then $F$ satisfies (cyc)$_\ell$.

We fix a coherent system $\zeta_{\ell\infty} := (\zeta_{\ell^n})_{n \geq 1}$ of $\ell$-power roots of unity in $F^{\text{sep}}$ and regard $\zeta_{\ell\infty}$ as a $\mathbb{Z}_\ell$-basis of $\mathbb{Z}_\ell/(1) = \lim_{\leftarrow} \mathbb{Z}_\ell/(\ell^n)$ where $\mu_{\ell^n} := \Spec(\mathbb{Z}[\ell]/(\ell^n - 1))$. By the condition (cyc)$_\ell$, two $\mathcal{G}_F$-modules $Q_\ell(m)$ and $Q_\ell(m')$ are not isomorphic for any two distinct integers $m$ and $m'$.

**Remark 4.2.** The condition (cyc)$_\ell$ is not equivalent to the condition that $\cup_{n \geq 1} \mu_{\ell^n}(F^{\text{sep}}) \subseteq F$. For example, the maximal totally real subfield $Q^{\text{tr}}(\mu_{\ell\infty})$ of $Q_\ell$ does not contain $\mu_{\ell\infty} := \cup_{n \geq 1} \mu_{\ell^n}(\overline{Q})$, although the order of the $\ell$-adic cyclotomic character on $\mathcal{G}_Q$ is two.

We denote by $\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F)$ the category of continuous representations of $\mathcal{G}_F$ on finite dimensional $\mathbb{Q}_\ell$-vector spaces. An $\ell$-adic mixed Tate $\mathcal{G}_F$-module is an object $V$ in $\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F)$ equipped with an increasing, saturated, and separated filtration $W_{2n}V = F^{\text{sep}}WV$ is a direct sum of $Q_\ell(−n)$ as a $\mathcal{G}_F$-module (cf. [12, Section 6, Section 7]). We denote by $\mathcal{MT}_\ell(F)$ the category of $\ell$-adic mixed Tate $\mathcal{G}_F$-modules. Then $\mathcal{MT}_\ell(F)$ is a mixed Tate category over $\mathbb{Q}_\ell$. The second example of an abstract modified polylogarithm is the $\ell$-adic étale modified polylogarithm. We apply our abstract formalism to $\mathcal{MT}_\ell(F)$.

4.1. Classifying spaces of torsors in $\mathcal{MT}_\ell(F)$. In this subsection, we make remarks on classifying spaces of torsors in $\mathcal{MT}_\ell(F)$.

**Lemma 4.3.** Let $(V_1, W_1 V_1)$ and $(V_2, W_2 V_2)$ be objects in $\mathcal{MT}_\ell(F)$. Let $f : V_1 \twoheadrightarrow V_2$ be an isomorphism of $Q_\ell[\mathcal{G}_F]$-modules. Then $f$ preserves the weight filtration of both-hand sides. In other words, $f$ defines a morphism in $\mathcal{MT}_\ell(F)$.

**Proof.** We show the lemma by induction on $l$ of the length of the weight filtration of $V_1$. If $l = 1$, then the assertion is clear. We suppose that the lemma holds if we replace $l$ by $l - 1$. For $i = 1, 2$, let $n_i$ be the minimal integer such that $W_{2n_i}V_i \neq 0$ and $W_{2n_i - 2}V_i = 0$. Since $f$ is an isomorphism of $Q_\ell[\mathcal{G}_F]$-modules, we have $n_1 = n_2$ and $f(W_{2n_1}V_1) = W_{2n_2}V_2$. We put $\overline{V}_i := V_i/W_{2n_i}V_i$ and denote by $\overline{f} : \overline{V}_1 \twoheadrightarrow \overline{V}_2$ the induced $Q_\ell[\mathcal{G}_F]$-homomorphism by $f$. Then, by induction hypothesis, $\overline{f}$ preserves the weight filtrations of both-hand sides. Since $W_{2n_i}V_i$ is the inverse image of $W_{2n_i}\overline{V}_i$ for each integer $n$ greater than $n_1 = n_2$, we have the conclusion of the lemma.

Let us fix an algebraic group $G = \text{Spec}(R)$ in $\mathcal{MT}_\ell(F)$ satisfying (Pos). Recall that $G(Q_\ell)$ has a natural topology on which $\mathcal{G}_F$ acts continuously (cf. [22, Section 3.2]).

**Lemma 4.4.** There exists a natural isomorphism of pointed sets

$$H^1(F, G(Q_\ell)) \cong H^1(\mathcal{MT}_\ell(F), G).$$

Here, the left-hand side is the first continuous Galois cohomology with coefficients in $G(Q_\ell)$.

**Proof.** According to Proposition 2.3, the underlying algebraic group of $\omega(G)$ is unipotent. Since $\omega$ and the forgetful functor

$$\mathcal{MT}_\ell(F) \to \text{Vec}_{\mathbb{Q}_\ell}(V, W_\bullet V) \to V$$

are locally isomorphic for the fpqc topology on $\text{Spec}(Q_\ell)$ (cf. [21, Theorem 3.2.3]), the underlying algebraic group of $G$ is also unipotent for the fpqc topology. This implies that $G$ itself is a...
unipotent algebraic group over $\mathbb{Q}_\ell$. Therefore the set of $\mathbb{Q}_\ell$-rational points of $G$-torsor is non-empty.

For each continuous 1-cocycle $c: \mathcal{G}_F \to G(\mathbb{Q}_\ell)$, we define the new action $a_c$ of $\mathcal{G}_F$ on $R = \mathcal{O}(G)$ by

$$a_c(\sigma)(f) := c(\sigma)^{(\sigma f)}$$

and define $R_c$ to be the ring $R$ equipped with the new action of $\mathcal{G}_F$ defined by $a_c$. Let us denote by $H^1(\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F), G)$ the set of isomorphism classes of torsors under $G$ in $\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F)$ and let $G_c := \text{Spec}(R_c)$ equipped with the natural right $G$-action. Then, since $X(\mathbb{Q}_p)$ is non-empty for any $G$-torsor $X$ in $\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F)$, we have a natural isomorphism

$$H^1(F, G(\mathbb{Q}_\ell)) \xrightarrow{\sim} H^1(\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F), G); [c] \mapsto [G_c]$$

(cf. [22, Proposition 3.15]). Therefore, to prove the lemma, it is sufficient to show that the natural map

$$H^1(MT_{\ell}(F), G) \to H^1(\text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F), G)$$

induced by the forgetful functor

$$MT_{\ell}(F) \to \text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F); (V, W • V) \mapsto V$$

is bijective. The injectivity of (4.2) follows from Lemma 4.3 directly. Hence, to show this lemma, it is sufficient to show the surjectivity of the map (4.2).

We put $W_{2n}R_c := W_{2n}R$. Then, according to Lemma A.4 and (4.1), the action of $\mathcal{G}_F$ on $R_c$ preserves the filtration $\{W_{2n}R_c\}_{n \in \mathbb{Z}}$ and coincides with the original action of $\mathcal{G}_F$ on each graded piece $\text{gr}_{2n}W_{2n}R_c$ of $R_c$. Therefore, the pair $(R_c, W_{2n}R_c)$ is an object in $\text{Ind}(MT_{\ell}(F))$ and this implies that (4.2) is surjective. □

If $G$ satisfies (Triv), then Lemma 4.4 can be rewritten as follows. Let $\mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$ be an isomorphism of $\mathbb{Q}_\ell$-vector spaces defined by $\zeta_{\ell^\infty}$. Then this isomorphism induces an isomorphism

$$\alpha: G = \text{Spec}(A) \xrightarrow{\sim} \omega(G) = \text{Spec}(\omega(A))$$

of algebraic groups over $\mathbb{Q}_\ell$ in the usual sense because $A$ splits into a direct sum of $\mathbb{Q}_\ell(n)$ by (Triv). Then, for each continuous 1-cocycle

$$c: \mathcal{G}_F \to G(\mathbb{Q}_\ell),$$

there exists a unique rational 1-cocycle

$$\tilde{c}: \pi_1(MT_{\ell}(F), \omega) \to \omega(G)$$

which fits into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G}_F \arrow{r}{\rho} \arrow{d}{c} & \pi_1(MT_{\ell}(F), \omega)(\mathbb{Q}_\ell) \arrow{d}{\tilde{c}} \\
G(\mathbb{Q}_\ell) \arrow{r}{\alpha} & \omega(G)(\mathbb{Q}_\ell).
\end{tikzcd}$$

Here, $\rho$ is the natural group homomorphism induced by the forgetful functor (4.3).

**Corollary 4.5** (cf. [12, Corollary 9.3]). For each positive integer $m$, there exists a natural isomorphism

$$\text{Ext}^1_{MT_{\ell}(F)}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(m)) \xrightarrow{\sim} H^1(F, \mathbb{Q}_\ell(m)).$$
Proof. Let \( A(Q_{\ell}(m)) := Sp(Sym^\bullet Q_{\ell}(-m)) \) be the vector group scheme in \( MT_\ell(F) \) defined by \( Q_{\ell}(m) \). Then, by a standard argument, \( Ext^1_{MT_\ell(F)}(Q_{\ell}(0), Q_{\ell}(m)) \) is canonically isomorphic to \( H^1(MT_\ell(F), A(Q_{\ell}(m))) \). Since \( m \) is positive, the group scheme \( A(Q_{\ell}(m)) \) satisfies (Pos). Hence, we have the conclusion of the corollary by Lemma 4.4. \( \square \) 

In later sections, we always identify \( Ext^1_{MT_\ell(F)}(Q_{\ell}(0), Q_{\ell}(m)) \) with \( H^1(F, Q_{\ell}(m)) \) for each positive integer \( m \).

4.2. Definition of \( \ell \)-adic étale modified polylogarithms

In this and the next subsections, we fix a positive integer \( m \).

**Definition 4.6.** We define the \( \ell \)-adic étale polylogarithmic quotient \( \mathcal{R}_m^\text{-ét} \), which is an algebraic group in \( MT_\ell(F) \), to be \( \mathcal{R}_m^\text{MT}_\ell(F) = \exp(Q_{\ell}(1) \ltimes \prod_{n=1}^m Q_{\ell}(n)) \).

We recall another description of \( \ell \)-adic étale polylogarithmic quotients (cf. [19, Section 2.1]). Let \( \pi_1^1(P_{01,\text{sep},F_{\text{sep}}}, \overline{0}) \) be the maximal pro-\( \ell \) quotient of the étale fundamental group of \( P_{01,\text{sep},F_{\text{sep}}} \) with the base point \( \overline{0} \). This pro-\( \ell \) group has a standard set of free generators \( \{ x, y \} \) (cf. [27, Section 8, Picture 4], [11, Exposé XIII, Corollaire 2.12]). Then, the group homomorphism

\[
p: \pi_1^1(P_{01,\text{sep},F_{\text{sep}}}, \overline{0}) \to \mathbb{Z}_\ell(1); x \mapsto \zeta_\ell, y \mapsto 1
\]

is \( G_F \)-equivariant where \( \zeta_\ell \) is the basis of \( \mathbb{Z}_\ell(1) \) fixed in the beginning of this section. We put

\[
\pi_{\text{pol}} := \pi_1^1(P_{01,\text{sep},F_{\text{sep}}}, \overline{0})/[\ker(p), \ker(p)]
\]

and

\[
\pi_{\text{pol}}(m) := \pi_{\text{pol}}/\pi_{\text{pol},(m)}
\]

where \( \{ \pi_{\text{pol},(n)} \}_{n=1}^\infty \) is the central descending series of \( \pi_{\text{pol}} \). Then, \( \mathcal{R}_m^\text{-ét} \) is canonically isomorphic to the unipotent completion \( \pi_{\text{pol}}(m)Q_{\ell} \) of \( \pi_{\text{pol}}(m) \) over \( Q_{\ell} \).

By the above remark, the algebraic groups \( \mathcal{R}_m^\text{-ét} \cong \pi_{\text{pol}}(m)Q_{\ell} \) are quotients of the unipotent completion \( \pi_{\text{pol}}(P_{01,\text{sep},F_{\text{sep}}}, \overline{0})Q_{\ell} \) of the étale fundamental group of \( P_{01,\text{sep},F_{\text{sep}}} \) with the base point \( \overline{0} \) over \( Q_{\ell} \) because of the functoriality of the unipotent completion. Let

\[
u_m^\text{-ét}: \pi_1^1(P_{01,\text{sep},F_{\text{sep}}}, \overline{0})Q_{\ell} \to \mathcal{R}_m^\text{-ét}
\]

be the canonical surjective homomorphism of affine group schemes in \( MT_\ell(F) \). For each \( z \in P_{01,\text{sep}}(F) \), we denote by \( \mathcal{R}_m^\text{-ét}(\overline{0}, z) \) the pushforward of the torsor \( \pi_1^1(P_{01,\text{sep},F_{\text{sep}}}, \overline{0})Q_{\ell} \) by \( \nu_m^\text{-ét} \).

**Definition 4.7.** We define the series of \( \ell \)-adic étale Albanese maps \( \text{Alb}^\ell_{F,m} = \{ \text{Alb}^\ell_{F,m} \} \) by

\[
\text{Alb}^\ell_{F,m}(z) := \mathcal{R}_m^\text{-ét}(\overline{0}, z) \text{ for all } z \in P_{01,\text{sep}}(F).
\]

It is well-known that \( \text{Alb}^\ell_{F,m} \) is a series of abstract Albanese maps (cf. [20, (3.1)]). Thus, we can define the \( \ell \)-adic étale modified polylogarithms as follows.

**Definition 4.8.** We define the \( m \)-th \( \ell \)-adic étale modified polylogarithm

\[
\mathcal{L}_m^\text{-ét}: \mathbb{Z}[P_{01,\text{sep}}(F)] \to \text{coLie}(MT_\ell(F))
\]

to be \( \mathcal{L}_m(\text{Alb}^\ell_{F,m}) \).
We define \( R_m^\ell(F), A_m^\ell(F), \) and \( B_m^\ell(F) \) by
\[
R_m^\ell(F) := R_m(F, \text{Alb}_F^{\ell}), \quad A_m^\ell(F) := A_m(F, \text{Alb}_F^{\ell}), \quad B_m^\ell(F) := B_m(F, \text{Alb}_F^{\ell}).
\]

In [20], we used the notation "\( \ell^-\)adic" instead of "\( \ell^-\)ét". By Proposition 2.11, \( \mathcal{L}_m^\ell \) induces a well-defined and injective group homomorphism
\[
\mathcal{L}_m^\ell: B_m^\ell(F) \hookrightarrow H^1(F, \mathbb{Q}_\ell(m)) = \text{Ext}^1_{\mathcal{M}_\ell(F)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(m)).
\]

**Remark 4.9.** By construction, the \( m \)-th \( \ell^-\)adic étale Albanese map \( \text{Alb}_{F,m}^{\ell} \) is functorial in \( F \). Hence \( \mathcal{L}_m^\ell \) is also functorial in \( F \).

### 4.3. Comparison with \( \ell^-\)adic polylogarithms

In this subsection, we compare \( \mathcal{L}_m^\ell \) with the Wojtkowiak \( \ell^-\)adic polylogarithms. For readers' convenience, we give a quick review of the \( \ell^-\)adic polylogarithm \( \ell_m(z, \gamma) \) attached to a \( \mathbb{Q}_\ell \)-path \( \gamma \) from \( 0\overline{1} \) to \( z \) in \( \mathbb{P}^1_{01; F,\text{sep}} \). Recall that a \( \mathbb{Q}_\ell \)-path in \( \mathbb{P}^1_{01; F,\text{sep}} \) is defined as an element of \( \pi_1(\mathbb{P}^1_{01; F,\text{sep}}, 0\overline{1}, z)\mathbb{Q}_\ell(\mathbb{Q}_\ell) \). Let \( \iota: \pi_1(\mathbb{P}^1_{01; F,\text{sep}}, 0\overline{1}) \to \mathbb{Q}_\ell(\mathbb{Q}_\ell) \) be the multiplicative embedding to non-commutative formal power series defined by
\[
\iota(x) := \exp(x), \quad \iota(y) := \exp(y).
\]

Let \( f_\gamma: \mathcal{G}_F \to \pi_1(\mathbb{P}^1_{01; F,\text{sep}}, 0\overline{1})\mathbb{Q}_\ell(\mathbb{Q}_\ell) \) be the 1-cocycle defined by \( f_\gamma(\sigma) := \gamma^{-1} \circ \sigma \gamma \). Then, the \( m \)-th \( \ell^-\)adic polylogarithm \( \ell_m(z, \gamma) \) is defined by
\[
\ell_m(z, \gamma)(\sigma) := (-1)^{m-1} \times \text{the coefficient of } \log(\iota(f_\gamma(\sigma))) \text{ at ad}(X)^{m-1}(Y)
\]
(cf. [28, Definition 11.0.1]).

**Lemma 4.10.** Let \( c_\gamma: \mathcal{G}_F \to \mathcal{P}_m^\ell(\mathbb{Q}_\ell) \) be the composite of \( f_\gamma \) and the canonical homomorphism \( \text{pr}_m: \pi_1(\mathbb{P}^1_{01; F,\text{sep}}, 0\overline{1})\mathbb{Q}_\ell(\mathbb{Q}_\ell) \to \pi^{\text{pol}}(m)\mathbb{Q}_\ell(\mathbb{Q}_\ell) \cong \mathcal{P}_m^\ell(\mathbb{Q}_\ell) \). Then, the image of the cohomology class \( [c_\gamma] \in H^1(F, \mathcal{P}_m^\ell(\mathbb{Q}_\ell)) \) under the natural isomorphism
\[
H^1(F, \mathcal{P}_m^\ell(\mathbb{Q}_\ell)) \cong H^1(\mathcal{MT}_\ell(F), \mathcal{P}_m^\ell)
\]
(cf. Lemma 4.4) coincides with \( \text{Alb}_{m,F}^{\ell}(z) \).

**Proof.** The cohomology class \( c_\gamma \) in \( H^1(F, \mathcal{P}_m^\ell(\mathbb{Q}_\ell)) \) represents the torsor under \( \mathcal{P}_m^\ell(\mathbb{Q}_\ell) \) in \( \text{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F) \) defined as the pushforward of \( \pi_1(\mathbb{P}^1_{01; F,\text{sep}}, 0\overline{1})\mathbb{Q}_\ell(\mathbb{Q}_\ell) \) by \( \text{pr}_m \). Therefore, the conclusion follows from the definition of the \( \ell^-\)adic étale Albanese map. \( \Box \)

Recall that the weight filtration \( W_{-2m} \mathcal{G}_F \) on \( \mathcal{G}_F \) is the filtration induced by the weight filtration on \( U(\mathcal{MT}_\ell(F)) \) (cf. [13, Section 7.4]). According to [13, Proposition 7.1, Lemma 7.5], we have canonical isomorphisms
\[
\text{gr}^{\text{W}}_{-2m} \mathcal{G}_F \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (\text{gr}^{\text{W}}_{-2m}U(\mathcal{MT}_\ell(F))) \mathbb{Q}_\ell \cong \text{Lie}(\mathcal{MT}_\ell(F))^{(-2m)}
\]
of \( \mathbb{Q}_\ell \)-vector spaces.

**Proposition 4.11.** Let \( \gamma \) be a \( \mathbb{Q}_\ell \)-path from \( 0\overline{1} \) to \( z \) in \( \mathbb{P}^1_{01; F,\text{sep}} \). The composite of homomorphisms
\[
W_{-2m} \mathcal{G}_F \to \text{gr}^{\text{W}}_{-2m} \mathcal{G}_F \hookrightarrow \text{Lie}(\mathcal{MT}_\ell(F))^{(-2m)} \xrightarrow{\mathcal{L}_m^{\ell}(z)} \mathbb{Q}_\ell
\]
coincides with the restriction of \( \ell_m(z, \gamma) \) to \( W_{-2m} \mathcal{G}_F \). In particular, that restriction does not depend on the choice of \( \gamma \).
Proof. Let \( c_\gamma : \mathcal{G}_F \to \mathcal{P}_m(Q_\ell) \) be the continuous 1-cocycle defined by \( \gamma \) (cf. Lemma 4.10). Then, there exists a unique rational 1-cocycle \( \tilde{c}_\gamma : \pi_1(M\mathcal{T}_{\ell}(F), \omega) \to \omega(\mathcal{P}_m) \) which fits into the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{G}_F & \xrightarrow{\rho} & \pi_1(M\mathcal{T}_{\ell}(F), \omega)(Q_\ell) \\
\downarrow{c_\gamma} & & \downarrow{\tilde{c}_\gamma} \\
\mathcal{P}_m^{\ell-\text{ét}}(Q_\ell) & \xrightarrow{\alpha} & \omega(\mathcal{P}_m^{\ell-\text{ét}}(Q_\ell)),
\end{array}
\]

where \( \alpha : \mathcal{P}_m^{\ell-\text{ét}} \xrightarrow{\sim} \omega(\mathcal{P}_m^{\ell-\text{ét}}) \) is an isomorphism of algebraic groups over \( Q_\ell \) induced by the fixed \( Q_\ell \)-basis \( \zeta_{\ell^\infty} \) of \( Q_\ell(1) \). Then, by Lemma 4.10 and the equation (2.4), we have

\[
\text{gr}^W_{-2m}(\tilde{c}_\gamma) = \mathcal{L}_m^{\ell-\text{ét}}(z)
\]
under the natural isomorphism

\[
((\text{gr}^W_{-2m}U(M\mathcal{T}_{\ell}(F)))(Q_\ell) \cong \text{Lie}(M\mathcal{T}_{\ell}(F))^{(-2m)}).
\]

Therefore, we have the following commutative diagram:

\[
\begin{array}{ccc}
W_{-2m}\mathcal{G}_F & \xrightarrow{\rho} & W_{-2m}U(M\mathcal{T}_{\ell}(F))(Q_\ell) \\
\downarrow{c_\gamma} & & \downarrow{\tilde{c}_\gamma} \\
Q_\ell(m) & \xrightarrow{c_{\gamma}^{\otimes m} \text{rep}} & \omega(Q_\ell(m)) = \mathcal{L}_m^{\ell-\text{ét}}(z)
\end{array}
\]

On the other hand, by the definition of the \( \ell \)-adic polylogarithms, we have the equality

\[
c_\gamma(\sigma) = \ell i_m(z, \gamma)(\sigma) \zeta_{\ell^m}
\]
for all \( \sigma \in W_{-2m}\mathcal{G}_F \). Hence we have the conclusion of the proposition. \( \square \)

**Lemma 4.12.** The restriction map

\[
H^1(F, Q_\ell(m)) \to H^1(W_{-2m}\mathcal{G}_F, Q_\ell(m))
\]
is injective.

**Proof.** By the Hochschild–Serre spectral sequence, it is sufficient to show the vanishing of the continuous group cohomology \( H^1(\mathcal{G}_F/W_{-2m}\mathcal{G}_F, Q_\ell(m)) \). By using the Hochschild–Serre spectral sequence again, we have the exact sequence

\[
0 \to H^1(\text{Gal}(F(\mu_{\ell^\infty})/F), Q_\ell(m)) \to H^1(\mathcal{G}_F/W_{-2m}\mathcal{G}_F, Q_\ell(m))
\to \text{Hom}^\text{cont}_{\text{Gal}(F(\mu_{\ell^\infty})/F)}(W_{-2m}\mathcal{G}_F/W_{-2m}\mathcal{G}_F, Q_\ell(m)).
\]

Since \( m \) is non-zero, the first term of (4.8) vanishes. On the other hand, \( \mathcal{G}_F \) acts on the graded quotient \( \text{gr}^W_{-2m}\mathcal{G}_F \) via the \( n \)-th power of the \( \ell \)-adic cyclotomic character by the definition of the weight filtration of \( \mathcal{G}_F \). Hence there exists no non-trivial \( \mathcal{G}_F \)-equivariant homomorphism from \( W_{-2}\mathcal{G}_F/W_{-2m}\mathcal{G}_F \) to \( Q_\ell(m) \). Thus the last term of (4.8) also vanishes and we have the conclusion of the lemma. \( \square \)

The following proposition is one of the keys of the proof of our main results Theorem A, Theorem B, and Theorem C.

**Proposition 4.13** (cf. [6, Theorem 2.3]). The image of \( B_{m}^{\ell-\text{ét}}(F) \otimes \mathbb{Z} Q \) under \( \mathcal{L}_m^{\ell-\text{ét}} \) coincides with \( [P_m^\ell(F)] \). In other words, a cohomology class \( x \in H^1(F, Q_\ell(m)) \) is represented by a linear sum

\[
\sum_i a_i \ell i_m(z_i, \gamma_i) : \mathcal{G}_F \to Q_\ell(m)
\]
with \( a_i \in \mathbb{Q} \), \( z_i \in F \), and \( \gamma_i : \overbar{Q} \rightarrow z_i \) if and only if \( x \) is contained in the image of \( B^\text{ét}_m(F) \otimes \mathbb{Z} \) under \( \mathcal{L}_m^\text{ét} \).

**Proof.** We first show the “if” part. Suppose that \( \xi \in A^\text{ét}_m(F) \). The existence of good paths \( \gamma_i \) follows by repeating exactly the same argument of the proof of [6, Theorem 2.3] by replacing the conjectural vector space \( \mathcal{L}_k \) in that paper by \((\mathbb{Z}[\mathbb{P}^{\infty}_{\delta_0}(F)]/\mathcal{R}_k^\text{ét}(F)) \otimes \mathbb{Z} \). Indeed, the latter vector space satisfies all the conditions that \( \mathcal{L}_k \) is expected to satisfy except the realization homomorphism \( B^\text{ét}_m(F) \rightarrow K_{2k-1}(F) \otimes \mathbb{Z} \) and it was not needed to the proof.

Next, we show the “only if” part. Let \( c : \mathcal{G}_F \rightarrow \mathbb{Q}_l(m) \) be a 1-cocycle of the form \( \sum_i a_i \ell_i m(z_i, \gamma_i) \). We put \( \xi := \sum_i a_i \ell_i m(z_i) \). Then, according to Proposition 4.11, the restriction of \( c \) to \( W_{-2m} \mathcal{G}_F \) coincides with the composite of \( \mathcal{L}_m^\text{ét}(\xi) \) and \( W_{-2m} \mathcal{G}_F \rightarrow \text{Lie}(\mathcal{M}_\ell(F))(-2m) \). Since the restriction \( c \) to \( \mathcal{G}_F(\mu_\infty) \) factors through the abelianization of \( \mathcal{G}_F(\mu_\infty) \), \( \mathcal{L}_m^\text{ét}(\xi) \) also factors through the abelianization homomorphism

\[
\text{Lie}(\mathcal{M}_\ell(F))(-2m) \subset \text{Lie}(\mathcal{M}_\ell(F)) \rightarrow \text{Lie}(\mathcal{M}_\ell(F))^{\text{ab}}.
\]

This implies that \( d \mathcal{L}_m^\text{ét}(\xi) = 0 \) and that \( \xi \in A^\text{ét}_m(F) \otimes \mathbb{Z} \) (cf. Remark 2.12). Since the restriction of the cohomology class \( \mathcal{L}_m^\text{ét}(\xi) \in H^1(F, \mathbb{Q}_l(m)) \) to \( W_{-2m} \mathcal{G}_F \) coincides with the restriction of the class of \( c \), \( \mathcal{L}_m^\text{ét}(\xi) \) agree with \( [c] \) in \( H^1(F, \mathbb{Q}_l(m)) \) by Lemma 4.12. This completes the proof of the proposition. \( \square \)

### 5. Motivic modified polylogarithms

The final example of an abstract modified polylogarithm is the motivic modified polylogarithm. Let \( k \) be a field of characteristic 0. In the end of the twenty century, M. Levine, M. Hanamura, and V. Voevodsky constructed the triangulated category of mixed motives independently. In the present paper, we use Voevodsky’s motives \( DM_{gm}(k) \) (cf. [25]). This triangulated category contains Tate objects \( \mathbb{Q}(n) \) for all integers \( n \). We denote by \( \mathcal{M}_\ell(k) \) the triangulated full-subcategory of \( DM_{gm}(k) \) generated by all Tate objects \( \mathbb{Q}(n) \). It is well-known that if the Beilinson–Soulé vanishing conjecture holds for \( k \), then \( \mathcal{M}_\ell(k) \) has a canonical t-structure in the sense of Beilinson–Bernstein–Deligne. The key of the existence of the category of mixed Tate motives over a number field \( K \) is the fact that the Beilinson–Soulé vanishing conjecture holds for \( K \). We let \( \mathcal{M}_\ell(K) \) be the heart of \( \mathcal{M}_\ell(K) \) and call an object in \( \mathcal{M}_\ell(K) \) a mixed Tate motive over \( K \).

Now, we fix a number field \( K \) for the rest of the present paper. We denote \( K_n(K) \otimes \mathbb{Z} \mathbb{Q} \) by \( K_n(K)_\mathbb{Q} \) where \( K_n(K) \) is the higher \( K \)-group of \( K \). For all integers \( n \) and \( m \), there exist the following canonical isomorphisms:

\[
\text{Hom}_{\mathcal{M}_\ell(K)}(\mathbb{Q}(n), \mathbb{Q}(m)) \cong \begin{cases} 0 & n \neq m, \\ \mathbb{Q} & n = m, \end{cases}
\]

\[(5.1)\]

\[
\text{Ext}^1_{\mathcal{M}_\ell(K)}(\mathbb{Q}(n), \mathbb{Q}(m)) \cong \begin{cases} 0 & n \geq m, \\ K_{2(m-n)-1}(K)_\mathbb{Q} & n < m, \end{cases}
\]

\[(5.2)\]

\[
\text{Ext}^2_{\mathcal{M}_\ell(K)}(\mathbb{Q}(n), \mathbb{Q}(m)) = 0.
\]

\[(5.3)\]

In particular, \( \mathcal{M}_\ell(K) \) is a mixed Tate category over \( \mathbb{Q} \). We always identify the left-hand sides of (5.1) and (5.2) with the right-hand sides of them by those canonical isomorphisms.

**Definition 5.1.** Let \( m \) be a non-negative integer. Then, we define \( \mathcal{H}_m^{\text{mot}} \), which is an algebraic group in \( \mathcal{M}_\ell(K) \), to be \( \mathcal{H}_m^{\mathcal{M}_\ell(K)} = \exp(\mathbb{Q}(1) \times \otimes_{n=1}^m \mathbb{Q}(n)) \).
It is known that the motivic polylogarithmic quotient is a quotient of the motivic fundamental group \( \pi_1^{\text{mot}}(\mathbb{P}^1_{\text{0\_\infty}, K}; \overline{01}) \) which is a pro-algebraic group in \( \mathcal{M}T(K) \) (cf. [5]). We denote by \( u_m^{\text{mot}} \) the canonical projection
\[
\pi_1^{\text{mot}}(\mathbb{P}^1_{\text{0\_\infty}, K}; \overline{01}) \to \mathcal{R}_m^{\text{mot}}.
\]

Deligne and Goncharov also constructed the path torsor \( \pi_1^{\text{mot}}(\mathbb{P}^1_{\text{0\_\infty}, K}; \overline{01}, z) \) from \( \overline{01} \) to \( z \) which is a right torsor under \( \pi_1^{\text{mot}}(\mathbb{P}^1_{\text{0\_\infty}, K}; \overline{01}) \) in \( \mathcal{M}T(K) \) for any \( K \)-rational base point \( z \) of \( \mathbb{P}^1_{\text{0\_\infty}, K} \) (cf. loc. cit.). We put
\[
\mathcal{R}_m^{\text{mot}}(\overline{01}, z) := u_m^{\text{mot}}\left( \pi_1^{\text{mot}}(\mathbb{P}^1_{\text{0\_\infty}, K}; \overline{01}, z) \right).
\]

By definition, \( \mathcal{R}_m^{\text{mot}}(\overline{01}, z) \) is a right \( \mathcal{R}_m^{\text{mot}} \)-torsor in \( \mathcal{M}T(K) \). Then, we define the motivic Albanese map as follows:

**Definition 5.2.** Let \( m \) be a non-negative integer. The series of motivic Albanese maps \( \text{Alb}_K^{\text{mot}} := \{ \text{Alb}_{K,m}^{\text{mot}} \} \) is defined by
\[
\text{Alb}_{K,m}^{\text{mot}}(z) := \mathcal{R}_m^{\text{mot}}(\overline{01}, z) \text{ for all } z \in \mathbb{P}^1_{\text{0\_\infty}}(K).
\]

The following lemma is a well-known fact.

**Lemma 5.3 (cf. [5, Lemma 5.12]).** The collection of motivic Albanese maps is a series of abstract Albanese maps. Furthermore, the restriction of the 0-th motivic Albanese map
\[
\text{Alb}_K^{\text{mot}}: \mathbb{P}^1_{\text{0\_\infty}}(K) = K^\times \setminus \{ 1 \} \to K_1(K)_{\mathbb{Q}} = K^\times \otimes_{\mathbb{Z}} \mathbb{Q}
\]
is the natural map induced by the identity of \( K^\times \setminus \{ 1 \} \).

**Definition 5.4.** We define the \( m \)-th motivic modified polylogarithm
\[
\mathcal{L}_m^{\text{mot}}: \mathbb{Z}[\mathbb{P}^1_{\text{0\_\infty}}(K)] \to \text{colie}(\mathcal{M}T(K))(2^m)
\]
to be \( \mathcal{L}_m(\text{Alb}_K^{\text{mot}}) \).

We define \( R_m^{\text{mot}}(K), A_m^{\text{mot}}(K), \) and \( B_m^{\text{mot}}(K) \) by
\[
R_m^{\text{mot}}(K) := R_m(K, \text{Alb}_K^{\text{mot}}), \quad A_m^{\text{mot}}(K) := A_m(K, \text{Alb}_K^{\text{mot}}), \quad B_m^{\text{mot}}(K) := B_m(K, \text{Alb}_K^{\text{mot}}),
\]
respectively. Then, the motivic modified polylogarithm induces an injective and well-defined homomorphism
\[
\mathcal{L}_m^{\text{mot}}: B_m^{\text{mot}}(K) \hookrightarrow \text{Ext}^1_{\mathcal{M}T(K)}(\mathbb{Q}(0), \mathbb{Q}(m)) = K_{2m-1}(K)_{\mathbb{Q}}.
\]

6. Proofs of Theorem A and Theorem B

6.1. Mixed realizations of motives. In this subsection, we recall mixed realizations of mixed Tate motives (cf. [4], [14], [15], [16], [5]). A. Huber had constructed the category \( D_{\text{MRS}} \) which contains the derived category of systems of realizations over a field \( k \) which can be embedded into \( \mathbb{C} \) (cf. [14, Definition 11.1.3]). The category \( D_{\text{MRS}} \) is equipped with the canonical functors
\[
\mathfrak{q}_\ell: D_{\text{MRS}} \to D(\text{Spec}(k)_{\ell}, \mathbb{Q}_\ell)
\]
and
\[
\mathfrak{q}_{H\ell}^+: D_{\text{MRS}} \to D(\text{MHS}(\mathbb{Q}))
\]
for each rational prime \( \ell \) and \( \sigma; k \hookrightarrow \mathbb{C} \). Here, \( D(\text{Spec}(k)_{\ell}, \mathbb{Q}_\ell) \) and \( D(\text{MHS}(k)) \) are the derived category of smooth \( \mathbb{Q}_\ell \)-sheaves on \( \text{Spec}(k)_{\ell} \) and mixed Hodge structures over \( \mathbb{Q} \), respectively. In [15] and [16], she also had constructed a functor between triangulated categories
\[
R_{\text{MRS}}: DM_{\text{gm}}(k) \to D_{\text{MRS}}
\]
called the mixed realization functor for any field \( k \) which can be embedded into \( \mathbb{C} \). Let \( \sigma : k \hookrightarrow \mathbb{C} \) be an embedding. We denote by \( R_{\text{Hdg}}^\sigma \) (resp. \( R_\ell \)) the composite of \( R_{MR} \) and

\[
D_{MR} \xrightarrow{q_{\text{Hdg}}} D(MHS(\mathbb{Q})) \to D(MHS(\mathbb{R})) \quad \left( \text{resp. } D_{MR} \xrightarrow{q_\ell} D(\text{Spec}(k)_{\text{ét}}, \mathbb{Q}_\ell) \right).
\]

Let \( \ell \) be a rational prime and let \( \sigma : K \hookrightarrow \mathbb{C} \) be a field embedding. By Huber’s construction, we have \( R_\ell(Q(m)) = Q_\ell(m) \) and \( R_{\text{Hdg}}^\sigma(Q(m)) = R(\mathbb{R}(m)) \) for each integer \( m \). Therefore, \( R_\ell \) and \( R_{\text{Hdg}}^\sigma \) induce canonical functors

\[
R_\ell : \mathcal{MT}(K) \to \mathcal{MT}_\ell(K)
\]

and

\[
R_{\text{Hdg}}^\sigma : \mathcal{MT}(K) \to \mathcal{H},
\]

respectively. Then, we denote by

\[
\tau_\ell : \text{Ext}^1_{\mathcal{MT}(K)}(Q(0), Q(m)) \to \text{Ext}^1_{\mathcal{MT}_\ell(K)}(Q_\ell(0), Q_\ell(m))
\]

and

\[
\tau_{\text{Hdg}}^\sigma : \text{Ext}^1_{\mathcal{MT}(K)}(Q(0), Q(m)) \to \text{Ext}^1_{\mathcal{H}}(R, R(\mathbb{R}(m)))
\]

the canonical homomorphisms induced by \( R_\ell \) and \( R_{\text{Hdg}}^\sigma \), respectively.

**Theorem 6.1** ([17, Proposition 5]). Under the canonical identifications

\[
\text{Ext}^1_{\mathcal{MT}(K)}(Q(0), Q(m)) = K_{2m-1}(K)Q
\]

and

\[
\text{Ext}^1_{\mathcal{MT}_\ell(K)}(Q_\ell(0), Q_\ell(m)) = H^1(K, Q_\ell(m)),
\]

\( \tau_\ell \) coincides with the \( \ell \)-adic higher regulator defined in [24]. In particular, \( \tau_\ell \) is injective whose image is a \( Q \)-lattice of \( H^1(K, Q_\ell(m)) \).

**Theorem 6.2** (cf. [5, 1.6]). Under the canonical identifications

\[
\text{Ext}^1_{\mathcal{MT}(K)}(Q(0), Q(m)) = K_{2m-1}(K)Q
\]

and

\[
\text{Ext}^1_{\mathcal{H}}(R(0), R(m)) = R(m-1),
\]

\( \tau_{\text{Hdg}}^\sigma \) coincides with the Beilinson regulator. In particular,

\[
\tau_{\text{Hdg}, K} := \bigoplus_{\sigma : K \hookrightarrow \mathbb{Q}} \tau_{\text{Hdg}}^\sigma : K_{2m-1}(K)Q \to \left( \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} R(m-1) \right)^{\text{Gal}(\mathbb{C}/\mathbb{R})}
\]

is injective whose image is a \( Q \)-lattice of the right-hand side.

### 6.2. Comparison of three modified polylogarithms.

Let \( z \) be a \( K \)-rational base point of \( \mathbb{P}^1_{01\infty,K} \). Then the tuple

\[
\mathcal{P}_m(z) := \left( \mathcal{P}_m^\text{Hdg}(01, \sigma(z)), \mathcal{P}_m^\text{ét}(01, \sigma(z)) \right)_{\sigma : K \hookrightarrow \mathbb{C}, \ell \text{ rational primes}}
\]

is a part of a scheme in the category of systems of realizations (cf. [5, 2.15]). Deligne and Goncharov proved the following theorem.

**Theorem 6.3** ([5, Théorème 4.4]). Let \( z \) be a \( K \)-rational base point of \( \mathbb{P}^1_{01\infty,K} \). Then, the system of realizations \( \mathcal{P}_m^\text{mot}(01, z) \) is motivic. More precisely, the Hodge realization attached to \( \sigma : K \hookrightarrow \mathbb{C} \) (resp. \( \ell \)-adic étale realization) of \( \mathcal{P}_m^\text{mot}(01, z) \) is canonically isomorphic to the torsor \( \mathcal{P}_m^\text{Hdg}(01, \sigma(z)) \) (resp. \( \mathcal{P}_m^\text{ét}(01, z) \)).
We denote by
\[ R'_{\text{Hdg},*} : \text{coLie}(MT(K)) \to \text{coLie}(H) \]
and by
\[ R_{\ell,*} : \text{coLie}(MT(K)) \to \text{coLie}(MT_{\ell}(K)) \]
the canonical coLie homomorphisms induced by \( R'_{\text{Hdg}} \) and \( R_{\ell} \), respectively. Remark that the restrictions of \( R'_{\text{Hdg},*} \) and \( R_{\ell,*} \) to \( K_{2m-1}(K)Q \subset \text{coLie}(MT(K))^{(2m)} \) coincide with \( r_{\text{Hdg}}' \) and \( r_{\ell} \), respectively.

**Proposition 6.4.** Let \( K \) be a number field and let \( m \) be a positive integer. Let \( \ell \) be a rational prime and let \( \sigma \) be an embedding \( K \hookrightarrow \mathbb{C} \). Then, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}[\mathbb{P}_{01\infty}^{\ell}(K)] & \xrightarrow{\mathcal{L}_{m}^{\text{mot}}} & \text{coLie}(MT(K))^{(2m)} \\
& \xrightarrow{\mathcal{L}_{m}^{\text{mot}}} & \text{coLie}(MT_{\ell}(K))^{(2m)} \\
& \xrightarrow{\mathcal{L}_{m}^{\text{mot}}} & \text{coLie}(H)^{(2m)} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}_{m}^{\text{Hdg} \circ \sigma} & \xrightarrow{R'_{\text{Hdg},*}} & \text{coLie}(H)^{(2m)} \\
& \xrightarrow{R_{\ell,*}} & \text{coLie}(MT_{\ell}(K))^{(2m)} \\
& & \text{coLie}(MT_{\ell}(K))^{(2m)}. \\
\end{array}
\]

**Proof.** The commutativity of the diagram is a direct consequence of Theorem 6.3 and constructions of modified polylogarithms. \( \square \)

**Corollary 6.5.** Let \( K \) be a number field and let \( \ell \) be a rational prime. Then, for each positive integer \( m \), we have equalities
\[ R_{m}^{\text{mot}}(K) = R_{m}^{\ell,\text{et}}(K) = R_{m}^{\text{Hdg}}(K). \]
In particular, for each positive integer \( m \), we have equalities of abelian groups
\[ A_{m}^{\text{mot}}(K) = A_{m}^{\ell,\text{et}}(K) = A_{m}^{\text{Hdg}}(K), \quad B_{m}^{\text{mot}}(K) = B_{m}^{\ell,\text{et}}(K) = B_{m}^{\text{Hdg}}(K). \]

**Proof.** Let \( \bullet \) be one of the symbols \( \text{Hdg}, \ell, \text{et}, \) and \( \text{mot} \). Since \( R_{m}^{\bullet}(K) \subset A_{m}^{\bullet}(K) \) and \( A_{m}^{\bullet}(K) \) is mapped to the first extension group by \( \mathcal{L}_{m}^{\bullet} \), it is sufficient to show the injectivity of the restrictions of \( R_{\ell,*} \) and \( \oplus_{\sigma} R'_{\text{Hdg},*} \) to \( K_{2m-1}(K)Q \). Theorem 6.1 and Theorem 6.2 guarantee those restrictions to be injective. \( \square \)

**Proof of Theorem C.** Let \( \xi = \sum a_{i} \{ z_{i} \} \) be an element of \( \mathbb{Q}[K \setminus \{0,1\}] \) such that the linear sum \( \sum_{i} a_{i} \mathcal{L}_{m}^{\text{mot}}(\sigma(z_{i})) \in R(m-1) \) is equal to 0 for all \( \sigma : K \hookrightarrow \mathbb{C} \). Then, by Corollary 6.5, \( \xi \) is contained in \( R_{m}^{\ell,\text{et}}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \). Hence, there exist specific \( \mathbb{Q}_{\ell}- \) paths \( \gamma_{i} : \overline{01} \to z_{i} \) such that \( \sum_{i} a_{i} \ell_{m}(z_{i}, \gamma_{i}) : \mathcal{G}_{K} \to \mathbb{Q}(m) \) is a 1-coboundary by Proposition 4.13. In particular, the restriction of \( \sum_{i} a_{i} \ell_{m}(z_{i}, \gamma_{i}) \) to \( \mathcal{G}_{K}(\mu_{\infty}) \) is identically zero. \( \square \)

### 6.3. Proofs of Theorem A and Theorem B.

**Proof of Theorem A.** According to Proposition 6.4 and Theorem 6.1, we have the following commutative diagram:

\[
\begin{array}{ccc}
B_{m}^{\text{mot}}(K) & \xrightarrow{\mathcal{L}_{m}^{\text{mot}}} & K_{2m-1}(K)Q \\
& \xrightarrow{\mathcal{L}_{m}^{\ell,\text{et}}} & H^{1}(K, \mathbb{Q}_{\ell}(m)) \\
& \xrightarrow{\text{reg}_{\ell,\text{et}}^{2m-1}} & \\
\end{array}
\]

\[
(6.1)
\]
where $\text{reg}_{\ell-\text{et}}$ is the Soulé $\ell$-adic higher regulator. Since $\text{reg}_{\ell-\text{et}}$ is injective, we have

\[(6.2) \quad \dim_{\mathbb{Q}}(\mathcal{L}_{m}^{\ell-\text{et}}(B_{m}^{\text{mot}}(K) \otimes \mathbb{Q})) \leq \dim_{\mathbb{Q}}(K_{2m-1}(K)_{\mathbb{Q}}).\]

On the other hand, it was proved by Borel that $\dim_{\mathbb{Q}}(K_{2m-1}(K)_{\mathbb{Q}}) = d_{m}$ for each positive integer $m$ greater than 1 (cf. [2]). Therefore, the inequality (1.3) of Conjecture 1.1 follows from (6.2), Corollary 6.5, and Proposition 4.13.

Next, we show that the inequality (1.3) is actually an equality when $m = 2, 3$. We have the following commutative diagram

\[(6.3) \quad \begin{array}{ccc}
B_{m}^{\text{mot}}(K) & \xrightarrow{\mathcal{L}_{m}^{\ell-\text{et}}} & K_{2m-1}(K)_{\mathbb{Q}} \\
\downarrow{\mathcal{L}_{m}^{\text{Hdg}}} & & \downarrow{\text{reg}_{\ell-\text{et}}^{\text{Hdg}}}
\end{array} \]

for all $\sigma : K \rightarrow \mathbb{C}$ by Proposition 6.4 and Theorem 6.2. According to Proposition 3.10, the restriction of $\mathcal{L}_{m}^{\text{Hdg}} \circ \sigma$ coincides with $\mathcal{L}_{m}^{\text{cl}} \circ \sigma$ multiplied by $-1$. On the other hand, if $m = 2$ or 3, then the map

$$-igoplus_{\sigma} \mathcal{L}_{m} \circ \sigma : B_{m}^{\text{mot}}(K) \otimes \mathbb{R} \rightarrow \left( \bigoplus_{\sigma : K \rightarrow \mathbb{C}} \mathbb{R}(m - 1) \right)^{\text{Gal}(\mathbb{C}/\mathbb{R})}$$

is surjective by the results of Zagier and Goncharov, respectively (cf. [29, Theorem 1], [8, Theorem 1.1]). This implies that the image of $B_{m}^{\text{mot}}(K)$ under $\mathcal{L}_{m}^{\text{mot}}$ is a full-lattice of $K_{2m-1}(K)_{\mathbb{Q}}$ when $m = 2, 3$. Therefore, the dimension of $B_{m}^{\ell-\text{et}}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = B_{m}^{\text{mot}}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with $d_{m}$ in those cases. This completes the proof of Theorem A. 

\[\square\]

**Proof of Theorem B.** Let $L$ be a finite Galois extension of $K$ with the Galois group $G$. Let $\xi = \sum a_{i}\{z_{i}\}$ be a $G$-invariant element of $\mathbb{Z} [P_{1}]^{I}(L)$ such that $c = \sum a_{i}^{\ell-\text{et}}(z_{i}, \gamma_{i})$ is a 1-cocycle for certain paths $\gamma_{i}$. According to Proposition 4.13, we have $\xi \in A_{m}^{\ell-\text{et}}(L)$ and $\mathcal{L}_{m}^{\ell-\text{et}}(\xi) = [c]$ in $H^{1}(L, Q_{m}(m))$. Then $[c]$ is also $G$-invariant because of the functoriality of $\mathcal{L}_{m}^{\ell-\text{et}}$ (cf. Remark 4.9). Similarly, $\mathcal{L}_{m}^{\text{mot}}(\xi)$ is also contained in the $G$-invariant part of $\text{Ext}^{1}_{\mathcal{MT}(L)}(Q(0), Q(m))$. Thus, the conclusion follows from the fact that the canonical homomorphism

$$\text{Ext}^{1}_{\mathcal{MT}(K)}(Q(0), Q(m)) \rightarrow \text{Ext}^{1}_{\mathcal{MT}(L)}(Q(0), Q(m))^{G}$$

is an isomorphism (cf. [5, Proposition 2.18]). 

\[\square\]

Finally, we give a conjecture stronger than Conjecture 1.1:

**Conjecture 6.6.** Let $K$ be a number field and let $\ell$ be a rational prime. Then we have $\dim_{\mathbb{Q}}[P_{m}^{\ell}(K)] = d_{m}$ for each positive integer $m$ greater than 1.

The following proposition guarantees the validity of Conjecture 6.6.

**Proposition 6.7.** Conjecture 6.6 is equivalent to the strong Zagier conjecture.

**Proof.** Recall that the strong Zagier conjecture states that $\mathcal{L}_{m}^{\text{mot}} : B_{m}^{\text{Hdg}}(K) \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow K_{2m-1}(K)_{\mathbb{Q}}$ is an isomorphism (cf. [30, p.417, Main Conjecture]). On the other hand, by Proposition 4.13, the $\mathbb{Q}$-vector space $[P_{m}^{\ell}(K)]$ is isomorphic to $B_{m}^{\ell-\text{et}}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $B_{m}^{\ell-\text{et}}(K) = B_{m}^{\text{Hdg}}(K)$ by Corollary 6.5, we have the conclusion of the proposition. 

\[\square\]
7. “Regulators” for $K_5$ via $\ell$-adic torilogarithms

Let $\ell$ be a rational prime and let $F$ be a field of characteristic $p \geq 0$ such that $p$ does not divide $\ell$. In the last of the present paper, we construct an explicit homomorphism from $K_5(F)$ to $H^1(F, \mathbb{Q}_\ell(3))$ via the $\ell$-adic trilogarithm. Then we show that this homomorphism coincides with the Soulé $\ell$-adic higher regulator multiplied by a non-zero rational number if $F$ is a number field. We use the following convention. For each $\xi$ with the Soulé $\ell$-adic higher regulator multiplied by a non-zero rational number if $F$ is a number field.

To construct the homomorphism, we recall works of Goncharov in [8] and [9]. Let $\tilde{C}_m(n)$ be the free abelian group generated by $m$-tuples of the $n$-dimensional $F$-vector space $F^n$ in generic position. Define $C_m(n)$ to be the maximal coinvariant quotient of $\tilde{C}_m(n)$ by the action of $\text{GL}_m(F)$. We call a $\text{GL}_m(F)$-orbit of an $m$-tuple of vectors an $m$-configuration of vectors of $F^n$. For an $m$-configuration $(v_1, \ldots, v_m)$ of vectors of $F^n$, we put

$$d(v_1, \ldots, v_m):= \sum_{i=1}^{m} (-1)^{i-1}(v_1, \ldots, \hat{v}_i, \ldots, v_m) \in C_{m-1}(n).$$

Then $\{C_m(n), d\}_m$ forms a complex of abelian groups. Let $C_\bullet(\text{GL}_m(F))$ be the standard free resolution of the $\text{GL}_m(F)$-module $\mathbb{Z}$. Then, according to [9, Lemma 3.1], there exists a natural homomorphism of complexes

$$(7.1) \quad g_n: C_\bullet(\text{GL}_m(F))_{\text{GL}_m(F)} \to C_\bullet(n),$$

where $C_\bullet(\text{GL}_m(F))_{\text{GL}_m(F)}$ is the maximal coinvariant quotient of $C_\bullet(\text{GL}_m(F))$ by $\text{GL}_m(F)$.

Now let us fix an isomorphism $\det(F^3) \cong F$. Goncharov introduced the linear map of abelian groups $f_6(3): \tilde{C}_6(3) \to \mathbb{Z}[\mathbb{P}^1(F)]$ defined by

$$(7.2) \quad f_6(3)(v_1, \ldots, v_6) := \text{Alt}_6 \left( \frac{\det(v_1, v_2, v_4) \det(v_2, v_3, v_5) \det(v_3, v_1, v_6)}{\det(v_1, v_2, v_5) \det(v_2, v_3, v_6) \det(v_3, v_1, v_4)} \right)$$

(cf. [9, Section 4, (9)]). Here, for an $F$-vector space $V$, a map $f: V^n \to \mathbb{P}^1(F)$, and $(v_1, \ldots, v_n) \in V^n$, $\text{Alt}_n(f(v_1, \ldots, v_n))$ is defined by

$$\text{Alt}_n(f(v_1, \ldots, v_n)) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \{ f(v_{\sigma(1)}, \ldots, v_{\sigma(n)} \} \in \mathbb{Z}[\mathbb{P}^1(F)].$$

It is easily checked that $f_6(3)$ does not depend on the choice of an isomorphism $\det(F^3) \cong F$. Let $R_n(F)$ be the Goncharov–Zagier abstract relation group (see [8, Section 1.9]) and let

$$\delta_3: \mathbb{Z}[\mathbb{P}^1(F)]/R_3(F) \otimes \mathbb{Q} \to \mathbb{Z}[\mathbb{P}^1(F)]/R_3(F) \otimes \mathbb{Q} F^\times \otimes \mathbb{Z} Q; \{ z \}_3 \mapsto \{ z \}_2 \otimes z,$$

where $\{ z \}_n$ is the image of $\{ z \}$ in $\mathbb{Z}[\mathbb{P}^1(F)]/R_n(F)$. Then Goncharov proved the following theorem:

**Theorem 7.1** (cf. [9, Section 3.2]).

1. The linear homomorphism $f_6(3)$ induces a group homomorphism

$$c_5'(3): H_5(C_\bullet(3) \otimes \mathbb{Q}) \to \text{Ker}(\delta_3).$$

2. Let $c_5(3): H_5(\text{GL}_3(F), \mathbb{Q}) \to \text{Ker}(\delta_3)$ be the composite of $c_5'(3)$ and the homomorphism $H_5(\text{GL}_3(F), \mathbb{Q}) \to H_5(C_\bullet(3) \otimes \mathbb{Q})$ induced by $g_3$ (see (7.1)). Then, for each $n \geq 3$, ...
there exists a commutative diagram

\[
\begin{array}{ccc}
H_5(\text{GL}_3(F), \mathbb{Q}) & \xrightarrow{c_5(3)} & \text{Ker}(\delta_3) \\
\downarrow c_5(n) & & \downarrow c_5(n) \\
H_5(\text{GL}_n(F), \mathbb{Q}) & & \\
\end{array}
\]

compatible with respect to \(n\) such that \(\text{Im}(c_5(3)) = \text{Im}(c_5(n))\).

In the subsequent paper [23], we will show the following lemma:

**Lemma 7.2.** Suppose that the field \(F\) satisfies the condition \((\text{cyc})_\ell\). Then, for any positive integer \(m\), the Zagier–Goncharov abstract relation group \(\mathcal{R}_m(F)\) is a subgroup of \(\langle R^\text{\acute{e}t}_{n}(F), \{0\}, \{\infty\} \rangle\).

By Lemma 7.2, we obtain a natural group homomorphism \(\text{Ker}(\delta_3) \rightarrow B_3^\text{\acute{e}t}(F) \otimes \mathbb{Z} \mathbb{Q}\) if \(F\) satisfies \((\text{cyc})_\ell\).

**Remark 7.3.** If \(F\) is a number field, then the assertion of Lemma 7.2 follows from Corollary 6.5 and [8, Corollary 1.19].

Now we give a new example of functional equation for the \(\ell\)-adic trilogarithm:

**Proposition 7.4** (Goncharov’s “7-terms” relation for \(\ell i_3\)). Let \(\ell\) be a rational prime and let \(F\) be a field of characteristic \(p \geq 0\) such that \(p \not|\ \ell\). Let \((v_0, \ldots, v_6)\) be a 7-tuple of vectors of \(F^3\) in generic position and put

\[\xi := \sum_{i=0}^{6} (-1)^i f_6(3)(v_0, \ldots, \tilde{v}_i, \ldots, v_6) \in \mathbb{Z}[\mathbb{P}^1(F)].\]

Then there exists a \(\mathbb{Q}_\ell\)-path system \(\gamma\) from \(\overrightarrow{01}\) to \(\xi\) in \(\mathbb{P}_{01, \infty, \text{sep}}^1\) such that the continuous function

\[\ell i_3(\xi, \gamma) : \mathcal{G}_F(\mu_\infty) \rightarrow \mathbb{Q}_\ell(3)\]

is identically zero.

**Proof.** Let \(k_p\) be the prime field of characteristic \(p\). Write \(\xi\) as \(\sum_{i=1}^{7} a_i \{z_i\} \in \mathbb{Z}[\mathbb{P}^1(F)]\). Then, by replacing \(F\) with \(k_p(z_1, \ldots, z_7)\), we may assume that \(F\) satisfies the condition \((\text{cyc})_\ell\) (cf. Example 4.1). By definition, the image of \(\sum_{i=0}^{6} (-1)^i (v_0, \ldots, \tilde{v}_i, \ldots, v_6)\) in the homology group \(H_5(C_\bullet(3) \otimes \mathbb{Z} \mathbb{Q})\) is equal to 0. Therefore, by Theorem 7.1 (1), the image of \(\xi\) in \(B_3^\text{\acute{e}t}(F)\) is also equal to 0. Therefore, by Proposition 4.13, there exists a system of \(\mathbb{Q}_\ell\)-path \(\gamma\) from \(\overrightarrow{01}\) to \(\xi\) such that \(\ell i_3(\xi, \gamma)\) is a 1-coboundary. Thus its restriction to \(\mathcal{G}_F(\mu_\infty)\) is identically zero. \(\square\)

If \(F\) satisfies the condition \((\text{cyc})_\ell\), then we obtain the following homomorphism compatible with respect to \(n \geq 3\):

\[r_{F,5,n} : H_5(\text{GL}_n(F), \mathbb{Z}) \rightarrow B_3^\text{\acute{e}t}(F) \otimes \mathbb{Z} \mathbb{Q} \rightarrow H^1(F, \mathbb{Q}_\ell(3)).\]

For a general field \(F\) of characteristic \(p\) such that \(p \not|\ \ell\), then we define the homomorphism \(r_{5,n}\) to be the injective limit

\[\lim_{\rightarrow} H_5(\text{GL}_n(F), \mathbb{Z}) = H_5(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{\lim_{\rightarrow} r_{F,5,n}} \lim_{\rightarrow} H^1(F, \mathbb{Q}_\ell(3)) \rightarrow H^1(F, \mathbb{Q}_\ell(3)).\]

Here, \(F_i\) runs over an injective system consisting of subfields of \(F\) satisfying \((\text{cyc})_\ell\) and the last map is the injective limit of the restriction maps.
DEFINITION 7.5. Let $\ell$ be a rational prime and let $F$ be a field of characteristic $p \geq 0$ such that $p \nmid \ell$. Then we define a group homomorphism

$$r_{5,\infty} : H_5(GL(F), \mathbb{Q}) \cong \lim_{n} H_5(GL_n(F), \mathbb{Q}) \to H^1(F, \mathbb{Q}_\ell(3))$$

to be the injective limit $\lim_{n} r_{5,n} \otimes \text{id}_\mathbb{Q}$. Here, $GL(F)$ is the injective limit $\lim_{n} GL_n(F)$ and the first isomorphism follows from the stability of the homology of $GL_n(F)$ proved by Quillen (cf. [26, Corollary]).

Recall that the Hurewitz map defines an injection from $K_5(F)_\mathbb{Q}$ to $H_5(GL(F), \mathbb{Q})$. Hence, $r_{5,\infty}$ defines a homomorphism from $K_5(F)_\mathbb{Q}$ to $H^1(F, \mathbb{Q}_\ell(3))$ and we use the same symbol $r_{5,\infty}$ for this homomorphism. By construction, the homomorphism

$$r_{5,\infty} : K_5(F)_\mathbb{Q} \to H^1(F, \mathbb{Q}_\ell(3))$$

can be written explicitly in the following sense. Let $v = (v_1, \ldots, v_6)$ be a 6-configuration of $F^3$. Then, for each element $g$ of $(g_1, \ldots, g_6) \in GL_3(F)^6$, we put $gv := (v_1g_1v_2, \ldots, g_5v_6)$. According to Theorem 7.1 (2), for any element $x$ of $K_5(F)_\mathbb{Q}$, there exists a 5-cycle $\sum_{j=1}^n b_j g_j$ of $GL_3(F)$ valued in $\mathbb{Q}$ such that $r_{5,3}(\sum_{j=1}^n b_j g_j) = r_{5,\infty}(x)$. Then we have

$$(7.3) \quad r_{5,\infty}(x) = \left[ \ell_3 \left( \sum_{j=1}^n b_j f_6(3)(g_j v), \gamma \right) \right] \in H^1(F, \mathbb{Q}_\ell(3)).$$

Here, $\gamma$ is a $\mathbb{Q}_\ell$-path system from $0 \overline{1}$ to $\sum_{j=1}^n b_j f_6(3)(g_j x)$ so that $\ell_3(\sum_{j=1}^n b_j f_6(3)(g_j x), \gamma)$ is a 1-cocycle.

THEOREM 7.6. If $F$ is a number field, then $r_{5,\infty}$ coincides with the Soulé $\ell$-adic higher regulator $\text{reg}_{\ell}^{\text{ét}}$ multiplied by a non-zero rational number.

Proof. By the commutative diagram (6.1), it is sufficient to show that the homomorphism

$$\tilde{c}_5 : K_5(F)_\mathbb{Q} \xrightarrow{\text{lim}_{n} c_{5}(n)} \text{Ker}(\delta_3) \to B_3^{\text{ét}}(F) \otimes \mathbb{Q} \text{ coincides with } (L_3^{\text{mot}})^{-1} \text{ multiplied by a non-zero rational number.}$$

Then, the conclusion of the theorem is a direct consequence of the following proposition. $\square$

PROPOSITION 7.7 (cf. [9, Theorem 5.11]). Let $F$ be a number field. Then the composite of $\tilde{c}_5$ and the homomorphism

$$L_3^{\text{Hdg}} : B_3^{\text{ét}}(F) \otimes \mathbb{Z} \mathbb{Q} = B_3^{\text{Hdg}}(F) \otimes \mathbb{Z} \mathbb{Q} \to \left( \bigoplus_{\sigma:F \to \mathbb{C}} \mathbb{R}(m-1) \right)^{\text{Gal(C/R)}}$$

coincides with the Borel regulator multiplied by a non-zero rational number.

By the similar method, we can construct $r_{3,\infty} : K_3(F) \to H^1(F, \mathbb{Q}_\ell(2))$ which had essentially constructed by Douai and Wojtkowiak in [6, p.84-85].

Appendix A. Group schemes in IFVec$_k$

In this appendix, we show useful lemmas for describing classifying spaces of torsors under algebraic groups in a mixed Tate category. Now, we fix a field $k$ and denote by IFVec$_k$ the category of finite dimensional $k$-vector spaces equipped with increasing, saturated, and separated filtrations. Denote by $(V, W_\bullet V)$ an object of IFVec$_k$ and we usually denote this object by $V$ for simplicity. Though this category is not an abelian category, we can consider group schemes in IFVec$_k$ as we did. In this appendix, we fix an algebraic group $G = \text{Sp}(R)$ in IFVec$_k$ satisfying the condition (Pos) (cf. Definition 2.2), namely, $R$ is a finitely generated Hopf algebra object in Ind(IFVec$_k$) satisfying $W_0R = k$. 
Lemma A.1. Let $V_1$, $V_2$, and $V_3$ be objects in $\text{Ind}($IFVec$_k$) and let $f: V_1 \to V_2 \otimes V_3$ be a morphism in $\text{Ind}($IFVec$_k$). For each $i = 1, 2, 3$, suppose that $W_nV_i = 0$ for all negative integers $n$. Then, for any $k$-linear homomorphism $g: V_3 \to k$, the composite of $k$-linear homomorphisms
\[(id_{V_2} \otimes g) \circ f: V_1 \xrightarrow{f} V_2 \otimes V_3 \xrightarrow{id_{V_2} \otimes g} V_2\]
is a morphism in $\text{Ind}($IFVec$_k$).

Proof. To prove the lemma, it is sufficient to show that $(id_{V_2} \otimes g) \circ f$ preserves the filtrations of both-hand sides. Since $W_nV_i$ vanish for all negative integers $n$, $f$ sends $W_nV_1$ to $\sum_{j,l \geq 0, j+l=n} W_jV_2 \otimes_k W_lV_3$. Therefore, $(id_{V_2} \otimes g) \circ f$ sends $W_nV_1$ to $\sum_{j,l \geq 0, j+l=n} W_jV_2$. Since $W_nV_i$ is an increasing filtration, we have $\sum_{j,l \geq 0, j+l=n} W_jV_2 = W_nV_2$. This completes the proof of the lemma.

The following corollaries are direct consequences of Lemma A.1.

Corollary A.2. Let $g$ be a $k$-rational point of the underlying algebraic group of $G$. Let $g^x: R \to R$ be the $k$-algebra automorphism induced by the left multiplication by $g$. Then the restriction of $g^x$ to $W_iR$ is an automorphism of $W_iR$ for each integer $i$.

Corollary A.3. Let $X = \text{Sp}(R^x)$ be an affine scheme in IFVec$_k$ equipped with a right action of $G$. Suppose that $W_nR^x = 0$ for all negative integer $n$. Then, for each $x \in X(k)$, the $k$-algebra homomorphism $x^*: R^x \to R$ induced by $G \to X; g \mapsto xg$ is a morphism in $\text{Ind}($IFVec$_k$).

Lemma A.4. Let $G = \text{Sp}(R)$ be an algebraic group in IFVec$_k$ satisfying (Pos). Let $g$ be a $k$-rational point of $G$. Then the automorphism on $\text{gr}_i^W R$ induced by $g^x$ is the identity map.

Proof. We denote by $e^x: R \to k$ the counit of $R$. Since $e^x$ is compatible with filtrations of both-hand sides, $\text{Ker}(e^x)$ is an object in $\text{Ind}($IFVec$_k$) such that $W_0(\text{Ker}(e^x)) = 0$. Let $c_{mr}$ be the comultiplication of $R$ and let $x$ be an element of $W_iR \setminus W_0R$. We write
\[c_{mr}(x) - 1 \otimes x = \sum_u a_u \otimes b_u \in W_0R \otimes_k W_iR + \sum_{j+l=i, j \geq 1, l \geq 0} W_j \text{Ker}(e^x) \otimes_k W_lR\]
such that $\{b_u\}_u$ are linearly independent over $k$. Remark that $c_{mr}(x) \neq 1 \otimes x$ because $x \notin k = W_0R$. Since $c_{mr}(x) - 1 \otimes x$ is contained in the kernel of $e^x \otimes 1$, all $a_u$ are contained in the kernel of $e^x$. This implies that all $b_u$ are contained in $W_{i-1}R$. Hence, we have $g^x(x) - x = \sum_i g(a_i)b_i \in W_{i-1}R$ and this implies that the induced $k$-linear homomorphism by $g^x$ on $\text{gr}_i^W R$ is the identity map.

Corollary A.5. Let $G = \text{Sp}(R)$ be an algebraic group in IFVec$_k$ satisfying (Pos). Then, the underlying algebraic group of $G$ is unipotent.

Proof. According to Lemma A.4, $W_iR$ is an algebraic representation of the underlying $k$-algebraic group $G$. Since $R = \bigcup W_iR$ and $G$ is of finite type, $W_nR$ is a faithful representation of $G$ for sufficiently large $N$. Moreover, $W_0R \subset W_1R \subset \cdots \subset W_NR$ is a flag of this representation and $G$ acts on all the graded quotients $\text{gr}_i^W R$ trivially. Hence, $G$ is unipotent.

The following lemma follows from the proof of Corollary A.5 easily, so we omit the proof of this lemma.

Lemma A.6. Let $G = \text{Sp}(R)$ be an algebraic group in IFVec$_k$ satisfying (Pos). Then, for each positive integer $N$, there exists a natural Lie homomorphism
\[\iota_N: \text{Lie}(G) \to \text{End}_k(W_NR); D \mapsto \log(\exp(D))^2|_{W_NR}\].

Furthermore, $\iota_N$ is injective for sufficiently large $N$. 

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