On functional equations of finite multiple polylogarithms

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Abstract

Recently, several people study finite multiple zeta values (FMZVs) and finite polylogarithms (FPs). In this paper, we introduce finite multiple polylogarithms (FMPs), which are natural generalizations of FMZVs and FPs, and we establish functional equations of FMPs. As applications of these functional equations, we calculate special values of FMPs containing generalizations of congruences obtained by Meštrović, Z. W. Sun, L. L. Zhao, Tauraso, and J. Zhao.

1. Introduction

From the end of twentieth century to the beginning of twenty-first century, Hoffman and J. Zhao had started research about mod \( p \) multiple harmonic sums, which are motivated by various generalizations of classical Wolstenholme’s theorem. Recently, Kaneko and Zagier introduced a new “adélic” framework to describe the pioneer works by Hoffman and Zhao and they defined finite multiple zeta values (FMZVs). Let \( k_1, \ldots, k_m \) be positive integers and \( \mathbf{k} := (k_1, \ldots, k_m) \).

**Definition 1.1** (Kaneko and Zagier [8, 9]). The finite multiple zeta value \( \zeta_\mathcal{A}(\mathbf{k}) \) is defined by

\[
\zeta_\mathcal{A}(\mathbf{k}) := \left( \sum_{\substack{p \nmid n_1 \cdots n_m \geq 0 \\mod p}} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \right) \in \mathcal{A}
\]

and the finite multiple zeta-star value \( \zeta_\mathcal{A}^*(\mathbf{k}) \) is defined by

\[
\zeta_\mathcal{A}^*(\mathbf{k}) := \left( \sum_{\substack{p - 1 \geq n_1 \geq \cdots \geq n_m \geq 1 \\mod p}} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \right) \in \mathcal{A}.
\]

Here, the \( \mathbb{Q} \)-algebra \( \mathcal{A} \) is defined by

\[
\mathcal{A} := \left( \prod_p \mathbb{F}_p \right) / \left( \bigoplus_p \mathbb{F}_p \right),
\]
where \( p \) runs over all prime numbers.

In this framework, Kaneko and Zagier established a conjecture, which states that there is an isomorphism between the \( \mathbb{Q} \)-algebra spanned by FMZVs and the quotient \( \mathbb{Q} \)-algebra modulo the ideal generated by \( \zeta(2) \) of the \( \mathbb{Q} \)-algebra spanned by the usual multiple zeta values.

On the other hand, Kontsevich [12], Elbaz-Vincent and Gangl [2] introduced finite version of polylogarithms and studied functional equations of them. Based on their works, Mattarei and Tauraso [16] calculated special values of finite polylogarithms.

Inspired by these studies, we introduce a finite version of multiple polylogarithms in the framework of Kaneko and Zagier:

**Definition 1.2** (See Definition 3.8). The finite multiple polylogarithms (FMPs) \( \mathcal{L}_{A,k}(t) \), \( \mathcal{L}^*_{A,k}(t) \), \( \widetilde{\mathcal{L}}_{A,k}(t) \), and \( \widetilde{\mathcal{L}}^*_{A,k}(t) \) are defined by

\[
\mathcal{L}_{A,k}(t) := \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t^{n_1} \cdots t^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)_{p \in A Z[t]},
\]

\[
\mathcal{L}^*_{A,k}(t) := \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{t^{n_1} \cdots t^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)_{p \in A Z[t]},
\]

\[
\widetilde{\mathcal{L}}_{A,k}(t) := \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t^{n_1} \cdots t^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)_{p \in A Z[t]},
\]

\[
\widetilde{\mathcal{L}}^*_{A,k}(t) := \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{t^{n_1} \cdots t^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)_{p \in A Z[t]}.
\]

Here, the \( \mathbb{Q} \)-algebra \( A_{Z[t]} \) is defined by

\[
A_{Z[t]} = \left( \prod_p \mathbb{F}_p[t] \right) / \left( \bigoplus_p \mathbb{F}_p[t] \right),
\]

where \( p \) runs over all prime numbers.

The symbol \( \mathcal{L} \) is used for the finite polylogarithms by Elbaz-Vincent and Gangl in their paper [2].

The main purpose of this paper is to establish functional equations of FMPs. Special cases of the main results are as follows:

**Theorem 1.3** (Main Theorem). The following functional equations hold in \( A_{Z[t]} \):

\[
(-1)^{m-1} \mathcal{L}_{A,k}(t) = \mathcal{L}^*_{A,k'}(1-t) - \zeta^*_A(k'),
\]

\[
(-1)^{m-1} \mathcal{L}_{A,k}(t) = \mathcal{L}^*_{A,k}(t) + \sum_{j=1}^{m-1} (-1)^j \mathcal{L}_{A,(k_1,\ldots,k_j)}(t) \zeta^*_A(k_m, \ldots, k_{j+1}).
\]

Here, \( k \) is the reverse index and \( k' \) the Hoffman dual of \( k \) (see Subsection 2.1).
The equality (1) is a generalization of the Hoffman duality [6, Theorem 4.6]. The
precise version of the main results are Theorem 3.12, Corollary 3.13, Remark 3.14, and
Theorem 3.15 which consist of the multiple variable cases of (1) and (2), the $A_2$-version
of (1), and the $A_n$-version of (2) for any positive integer $n$ (see Theorem 1.5 below and
Subsection 3.1 for the definition of $A_n$). Main results are obtained as mod $p$ reductions
of generalizations (= Theorem 2.5 and Theorem 2.10) of classical Euler’s identity:

$$\sum_{n=1}^{N} (-1)^{n-1} \binom{N}{n} \frac{1}{n} = \sum_{n=1}^{N} \frac{1}{n},$$

where $N$ is a positive integer ([3]). The functional equation (2) and its generalization also
hold for the usual multiple polylogarithms (Theorem 2.13).

Independently of us, Ono and Yamamoto gave another definition of FMPs and estab-
lished the shuffle relation in their preprint [18]. We will investigate a relation between our
FMPs and Ono-Yamamoto’s FMPs.

As applications of the functional equations, we will calculate some special values of the
finite multiple polylogarithms by using Tauraso and J. Zhao’s results for the alternating
multiple harmonic sums in Section 4.

Several people study supercongruences involving the harmonic numbers (Z. W. Sun,
L. L. Zhao, Medstrovic, and so on). For instance, Z. W. Sun and L. L. Zhao proved the
following congruence:

**Theorem 1.4** (Z. W. Sun and L. L. Zhao [23, Theorem 1.1]). Let $p$ be a prime number
greater than 3. Then

$$\sum_{k=1}^{p-1} \frac{H_k}{k^{2k}} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2},$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the $k$-th harmonic number and $B_{p-3}$ is the $(p-3)$-rd Bernoulli
number.

We can regard such congruences as explicit formulas of special values of FMPs and we
will give generalizations of some of them. As an application of main results, we obtain
the following theorem which is a generalization of Theorem 1.4 (= the case $m = 2$ in
Theorem 1.5):

**Theorem 1.5** (cf. Theorem 4.5). Let $m$ be an even positive integer. Then we have

$$\ell_{A_2, \{1\}^m}^* (1/2) = \left( \frac{2^{m+1} - 1}{2^{m+1}} B_{p-m-1} \frac{p \mod p^2}{m+1} \right)_p \text{ in } A_2,$$

where $B_{p-m-1}$ is the $(p-m-1)$-st Bernoulli number and

$$\ell_{A_2, \{1\}^m}^* (1/2) := \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \cdots n_m 2^{n_1}} \mod p^2 \right)_p \text{ in } A_2.$$

Here, the $\mathbb{Q}$-algebra $A_2$ is defined by

$$A_2 := \left( \prod_p \mathbb{Z}/p^2 \mathbb{Z} \right) \bigg/ \left( \bigoplus_p \mathbb{Z}/p^2 \mathbb{Z} \right).$$
This paper is organized as follows:
In Section 2, we prove generalizations of Euler’s identity involving binomial coefficients by introducing some formal truncated integral operators. In Section 3, we define the ring $A_{n,R}^\Sigma$ and recall some known results on FMZVs. Then we introduce FMPs and prove the functional equations by applying identities obtained in Section 2. We also study the relation between our FMPs and the ones defined by Ono and Yamamoto. In Section 4, as applications of the functional equations, we calculate special values of FMPs.

2. Generalizations of Euler’s identity

2.1. Notations for indices and the Hoffman dual

Here, we recall an involution introduced by Hoffman on the set of indices.

We define the set $I$ by

$$I := \coprod_{m \in \mathbb{Z}_{\geq 0}} (\mathbb{Z}_{\geq 0} \times \cdots \times \mathbb{Z}_{\geq 0})$$

and we call an element of $I$ an index. For an index $k = (k_1, \ldots, k_m) \in I$ we define the weight (resp. the depth) of $k$ to be $k_1 + \cdots + k_m$ (resp. $m$) and we denote it by $\text{wt}(k)$ (resp. $\text{dep}(k)$).

For a non-negative integer $k$, the symbol $\{k\}^m$ denotes $m$ repetitions $(k, \ldots, k) \in \mathbb{Z}_{\geq 0}^m$ of $k$. Let $e_i := (\{0\}^{i-1}, 1, \{0\}^{m-i})$ when $m$ is clear from the context. Throughout this paper, we use the following operators for indices:

For three indices $k = (k_1, \ldots, k_m), k_1 = (k'_1, \ldots, k'_m)$, and $k_2 = (k''_1, \ldots, k''_m)$, we define $\overline{k}, k_1 \sqcup k_2$, and $k \oplus k_1$ by $\overline{k} := (k_m, \ldots, k_1), k_1 \sqcup k_2 := (k'_1, \ldots, k'_m, k''_1, \ldots, k''_m)$, and $k \oplus k_1 := (k_1 + k'_1, \ldots, k_m + k''_m)$, respectively.

Let $W$ be the free monoid generated by the set $\{0, 1\}$. We denote by $W_1$ the set of words in $W$ of the form $\cdots 1$. We see that the correspondence

$$(k_1, \ldots, k_m) \mapsto \underbrace{0 \cdots 0}_{k_1-1} \underbrace{1 \cdots 0}_{k_2-1} \underbrace{1 \cdots 1}_{k_m-1}$$

induces a bijection $w : I \rightarrow W_1$.

**Definition 2.1** (cf. [6, Section 3]). Let $\tau : W \rightarrow W$ be a monoid homomorphism defined by $\tau(0) = 1$ and $\tau(1) = 0$. Then we define an involution $\tau : I \rightarrow I$ by the equality $\tau(\text{wt}(k)) = \text{wt}(\tau(k)) - 1$. We call this involution the Hoffman dual.

By the definition of the Hoffman dual, we see that $k^\tau = \overline{k}$ holds for any index $k$. We can use the notation $k^\tau$ since the Hoffman dual and the reversal operator $k \mapsto \overline{k}$ commute.

**Example 2.2.** We have the following equalities:

$$m^\tau = \{1\}^m, \quad (k_1, k_2)^\tau = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}),$$

$$(k_1, k_2, k_3)^\tau = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-2}, 2, \{1\}^{k_3-1}),$$

$$(k_1, \{1\}^{k_2-1})^\tau = (\{1\}^{k_1-1}, k_2).$$

Here, $m, k_1, k_2$ and $k_3$ are positive integers and the third equality holds when $k_2$ is greater than or equal to 2.
The following lemma is useful for inductive arguments on weight:

**Lemma 2.3.** Let $k$ be an index and $k^\vee$ its dual. Then we have $(k \oplus e_1)^\vee = \{1\} \cup k^\vee$ and $(\{1\} \sqcup k)^\vee = (k^\vee \oplus e_1)$.

**Proof.** These are obvious by the definition of the Hoffman dual. \square

We can prove that the equalities $\text{wt}(k^\vee) = \text{wt}(k)$ and $\text{dep}(k) + \text{dep}(k^\vee) = \text{wt}(k) + 1$ hold for any index $k$ by using the above lemma.

**Remark 2.4.** Hoffman defined the dual $k \mapsto k^\vee$ by another way (see [6, Section 3]). Let $I_w := \{k \in I \mid \text{wt}(k) = w\}$ for a positive integer $w$ and $\mathcal{P}(\{1, 2, \ldots, w - 1\})$ the power set of $\{1, 2, \ldots, w - 1\}$. Then there is a bijection $\psi: I_w \overset{\sim}{\to} \mathcal{P}(\{1, 2, \ldots, w - 1\})$ defined by the correspondence

$$k = (k_1, \ldots, k_m) \mapsto \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{m-1}\}$$

and the original Hoffman dual $k^\vee$ is defined by

$$k^\vee := \psi^{-1}(\{1, 2, \ldots, w - 1\} \setminus \psi(k))$$

for $k \in I_w$. This definition is equivalent to the first definition since the identities in Lemma 2.3 also hold for this dual.

### 2.2. Generalizations of Euler’s identity and their corollaries

In this subsection, we state polynomial identities which will be used for the proof of our main results in Subsection 3.2. We will give the proofs of Theorem 2.5 and Theorem 2.10 in Subsection 2.4. Through this subsection, let $R$ be a commutative ring including the field of rational numbers.

**Theorem 2.5.** Let $k = (k_1, \ldots, k_m)$ be an index of weight $w$ and $N$ a positive integer. Then the following polynomial identity holds in $R[t_1, \ldots, t_m]$:

$$\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1 \cdots n_w} =$$

$$\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (1 - t_1)^{n_1-n_1+1} \cdots (1 - t_{m-1})^{n_{m-1}-n_m+1} \frac{(1 - t_m)^{n_m} - 1}{n_1 \cdots n_w},$$

$$\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1 \cdots n_w} =$$

$$\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (1 - t_1)^{n_1-n_1+1} \cdots (1 - t_{m-1})^{n_{m-1}-n_m+1} \frac{(1 - t_m)^{n_m} - 1}{n_1 \cdots n_w}$$

where $l_1 = k_1, l_2 = k_1 + k_2, \ldots, l_m = k_1 + \cdots + k_m (= w)$.

When we substitute 1 for some of $t_1, \ldots, t_{m-1}$ in the left hand side of (4), some terms vanish and some $0^0$ appear in the summation of the right hand side. We consider $0^0$ as 1 since $t^0 = 1$ is a constant function for a variable $t$ and a positive integer $n$. In such situations, we can rewrite (4) as follows:
Hence we have the desired formula.

\[
\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \bigg|_{t_i=\cdots=t_h=1} = \sum_{N \geq n_1 \geq \cdots \geq n_m' \geq 1} \frac{(1 - t_1)^{n_1} (1 - t_2)^{n_2} \cdots (1 - t_{m-1})^{n_{m-1}} (1 - t_m)^{n_m - 1}}{n_1^{k_1} \cdots n_m^{k_m}} \prod_{j=1}^{h'} \left(1 - t_j\right)^{n_j} \left(1 - t_{j+h'}\right)^{n_{j+h'}} \left(1 - t_m\right)^{n_m' - 1},
\]

where \( L_i := l_{ji} - j_i + i \) for \( i = 1, \ldots, m' \).

**Proof.** If the condition

\[ n_{i_1} = n_{i_1+1}, \; n_{i_2} = n_{i_2+1}, \ldots, \; n_{i_h} = n_{i_h+1} \tag{\ast} \]

holds, then we have

\[
(1 - t_1)^{n_1} (1 - t_2)^{n_2} \cdots (1 - t_{m-1})^{n_{m-1}} (1 - t_m)^{n_m - 1} \prod_{j=1}^{h'} \left(1 - t_j\right)^{n_j} \left(1 - t_{j+h'}\right)^{n_{j+h'}} \left(1 - t_m\right)^{n_m' - 1}
\]

as a polynomial equality. On the other hand, if \( (n_1, \ldots, n_w) \) does not satisfy the condition (\( \ast \)), then the term for \( (n_1, \ldots, n_w) \) of the right hand side of Theorem 2.5 (4) vanishes when \( t_{i_1} = \cdots = t_{i_h} = 1 \).

By the definition of the Hoffman dual, we have

\[
\{l'_1, l'_2, \ldots, l'_{m'-1}\} = \{1, \ldots, w - 1\} \setminus \{l_{i_1}, \ldots, l_{i_h}\}
\]

where \( l'_1 = k'_1, l'_2 = k'_1 + k'_2, \ldots, l'_{m'-1} = k'_1 + \cdots + k'_{m'-1} \). Therefore we can rewrite the condition (\( \ast \)) as follows:

\[
n_1 = \cdots = n'_1, \; n'_1 + 1 = \cdots = n'_2, \ldots, \; n'_{m'-1} + 1 = \cdots = n_w.
\]

Hence we have the desired formula. \( \square \)

In particular, we have the following corollary:

**Corollary 2.7.** Let \( k = (k_1, \ldots, k_m) \) be an index, \( k^\vee = (k'_1, \ldots, k'_{m'}) \), and \( N \) a positive integer. Then we have the polynomial identity

\[
\sum_{N \geq n_1 \geq \cdots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} = \sum_{N \geq n_1 \geq \cdots \geq n_m' \geq 1} \frac{(1 - t)^{n_m'} - 1}{n_1^{k'_1} \cdots n_m^{k'_m}}, \tag{6}
\]

in \( R[t] \).

**Remark 2.8.** The case \( k = m \in \mathbb{Z}_{>0} \) of Corollary 2.7 (6) gives Tauraso-Zhao’s identity [25, Lemma 5.5 (42)] and the case \( t = 1 \) of Corollary 2.7 (6) gives Hoffman’s identity [6, Theorem 4.2]. Dilcher’s identity [1] and Hernández’s identity [4] are special cases of Hoffman’s identity. All these are generalizations of Euler’s identity (3).
Remark 2.9. Theorem 2.5 is also deduced from Kawashima-Tanaka’s formula [11, Theorem 2.6], which is a generalization of the identity

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \frac{1}{n + 1} = \frac{1}{N + 1}$$

(cf. [4, Woord’s solution]). Our proof of Theorem 2.5 in Subsection 2.4 is quite different from the proof by Kawashima and Tanaka.

**Theorem 2.10.** Let \( k = (k_1, \ldots, k_m) \) be an index of weight \( w \) and \( N \) a positive integer. Then the following identity holds in \( R[t_1, t_2^{\pm 1}, \ldots, t_m^{\pm 1}] \):

$$\sum_{N+1>n_1>\cdots>n_m>0} (-1)^{n_m} \binom{N}{n_m} \frac{(t_1/t_2)^{n_1} \cdots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

$$= \sum_{j=1}^{m-1} (-1)^{m-j-1} \left( \sum_{N+1>n_1>\cdots>n_j>0} \binom{N}{n_j} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \times$$

$$\left( \sum_{N+1>n_1>\cdots>n_{m-j}>0} \frac{(1-t_m)^{n_1-n_{i1}+1} \cdots (1-t_{j+2})^{n_{m-j}+1-n_{m-j}+1} \{(1-t_{j+1})^{n_{m-j}}-1\}}{n_1^{k_1} \cdots n_{m-j}^{k_{m-j}}} \right)$$

(7)

and

$$\sum_{N+1>n_1>\cdots>n_m>0} \frac{(t_1/t_2)^{n_1} \cdots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

$$= \sum_{j=1}^{m-1} (-1)^{m-j-1} \left( \sum_{N+1>n_1>\cdots>n_j>0} \binom{N}{n_j} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \times$$

$$\left( \sum_{N+1>n_1>\cdots>n_{m-j}>0} \frac{(1-t_m)^{n_1-n_{i1}+1} \cdots (1-t_{j+2})^{n_{m-j}+1-n_{m-j}+1} \{(1-t_{j+1})^{n_{m-j}}-1\}}{n_1^{k_1} \cdots n_{m-j}^{k_{m-j}}} \right),$$

(8)

where \( l_1 = k_m, l_2 = k_m + k_{m-1}, \ldots, l_m = k_m + \cdots + k_1 (= w) \).

**Theorem 2.11.** Let \( k = (k_1, \ldots, k_m) \) be an index and \( N \) a positive integer. Then the following identity holds in \( R[t_1, \ldots, t_m] \):

$$(-1)^{m-1} \sum_{N+1>n_1>\cdots>n_m>0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} = \sum_{N+1>n_1>\cdots>n_m>0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

$$+ \sum_{j=1}^{m-1} (-1)^j \left( \sum_{N+1>n_1>\cdots>n_j>0} \frac{t_1^{n_1} \cdots t_j^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \left( \sum_{N+1>n_1>\cdots>n_{m-j}>0} \frac{t_1^{n_1} \cdots t_{j+1}^{n_{j+1}}}{n_1^{k_1} \cdots n_{m-j}^{k_{m-j}}} \right).$$
Proof. By combining Theorem 2.5 (5) and Theorem 2.10 (8), we have

\[ (-1)^{m-1} \sum_{N+1>n_1>\cdots>n_m>0} \frac{(t_1/t_2)^{n_1} \cdots (t_{m-1}/t_m)^{n_{m-1}}}{n_1^{k_1} \cdots n_m^{k_m}} + \sum_{j=1}^{m-1} (-1)^j \left( \sum_{N+1>n_2>\cdots>n_j>0} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \left( \sum_{N+1>n_{j+1}>\cdots>n_m>0} \frac{(t_{j+1}/t_{j+2})^{n_{j+1}} \cdots (t_m)^{n_m}}{n_{j+1}^{k_{j+1}} \cdots n_m^{k_m}} \right) \]

in \( \mathbb{R}[t_1, t_2^{\pm 1}, \ldots, t_m^{\pm 1}] \). By replacing \( t_1/t_2 \mapsto t_1, \ldots, t_{m-1}/t_m \mapsto t_{m-1} \), we obtain the desired identity. \( \square \)

As applications of Theorem 2.11, we can prove not only a formula for the finite multiple polylogarithms (= Theorem 3.15) but also a formula for the usual multiple polylogarithms by taking the limit \( N \to \infty \). We recall the definition of the multiple polylogarithms:

**Definition 2.12.** Let \( k = (k_1, \ldots, k_m) \) be an index and \( z_1, \ldots, z_m \) complex numbers satisfying at least one of the following conditions for absolute convergence:

(i) \(|z_i| < 1 \) and \(|z_i| \leq 1 \) (\( 2 \leq i \leq m \)),

(ii) \(|z_i| \leq 1 \) (\( 1 \leq i \leq m \)) and \( k_1 \geq 2 \).

Then, we define the multiple polylogarithms by

\[ \text{Li}_k(z_1, \ldots, z_m) := \sum_{n_1>n_2>\cdots>n_m \geq 1} \frac{z_1^{n_1} \cdots z_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}, \quad \text{Li}_k^*(z_1, \ldots, z_m) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} \frac{z_1^{n_1} \cdots z_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}. \]

If \( k_1 \geq 2 \), then we define the multiple zeta(-star) values \( \zeta(k) \) and \( \zeta^*(k) \) by \( \zeta(k) := \text{Li}_k(\{1\}^m) \) and \( \zeta^*(k) := \text{Li}_k^*(\{1\}^m) \), respectively.

**Theorem 2.13.** Let \( k = (k_1, \ldots, k_m) \) be an index and \( z_1, \ldots, z_m \) complex numbers satisfying at least one of the following conditions:

(i) \(|z_1| < 1 \), \(|z_i| \leq 1 \) (\( 2 \leq i \leq m-1 \)), and \(|z_m| < 1 \),

(ii) \(|z_i| \leq 1 \) (\( 1 \leq i \leq m \)), \( k_1 \geq 2 \), and \( k_m \geq 2 \).

Then we have

\[ \sum_{j=0}^{m} (-1)^j \text{Li}_{(k_1, \ldots, k_j)}(z_1, \ldots, z_j) \text{Li}_{(k_{m-j+1}, \ldots, k_m)}^*(z_{m-j+1}, \ldots, z_m) = 0. \]

Here, we consider \( \text{Li}_{(k_1, \ldots, k_j)}(z_1, \ldots, z_j) \) (resp. \( \text{Li}_{(k_{m-j+1}, \ldots, k_m)}^*(z_{m-j+1}, \ldots, z_m) \)) as 1 when \( j = 0 \) (resp. \( j = m \)).

If \( k_1 \geq 2 \) and \( k_m \geq 2 \), then we have

\[ \sum_{j=0}^{m} (-1)^j \zeta(k_1, \ldots, k_j) \zeta^*(k_{m-j+1}, \ldots, k_m) = 0. \] (9)

See Remark 3.17.
2.3. Truncated integral operators

In this subsection, we introduce truncated integral operators to prove the theorems in Subsection 2.2. Through this subsection, let $R$ be a commutative ring including the field of rational numbers $\mathbb{Q}$ and $N$ a positive integer. Let $t$ and $s$ be indeterminates.

Let $\int s \, dt: R[t] \to R[t]$ be the formal indefinite integral operator satisfying the condition that the constant term with respect to $t$ is equal to 0, that is,

$$\int \left( \sum_{n=0}^{\infty} a_n t^n \right) \, dt := \sum_{n=0}^{\infty} \frac{a_n}{n+1} t^{n+1}.$$ 

We prepare the following five $R$-linear operators:

$$I_{t,s;R}: R[t,s] \to R[s/t][t], \quad I_{t,s;R}: R[t] \to R[s/t][t], \quad \tau_{t;R}^{\leq N}: R[t] \to R[t],$$

$$\text{pr}_{t;R}: R((t^{-1})) \to R[t], \quad f(t) \mapsto \int \frac{f(t)}{t} \, dt, \quad f(t,s) \mapsto \int \frac{f(t,s)}{t-s} \, ds,$$

$$\sum_{n=-\infty}^{n_0} a_n t^n \mapsto \begin{cases} \sum_{n=0}^{n_0} a_n t^n & \text{if } n_0 \text{ is non-negative}, \\ 0 & \text{otherwise}, \end{cases} \quad \sum_{n=n_0}^{\infty} a_n t^n \to \sum_{n=1}^{\infty} a_n t^{-n}. $$

Here, we consider the formal integral operator in the definition of $I_{t,s;R}$ as an operator on $R((t^{-1}))[s]$. For instance, we have

$$I_{t,s;R}(s^n) = \sum_{j=1}^{\infty} \frac{s^{n+j} t^{-j}}{j}, \quad (10)$$

for a non-negative integer $n$.

**Definition 2.14.** We define the truncated integral operators $J_{t,s;R}^*$ and $J_{t,s;R}^N$ by

$$J_{t,s;R}^*: \text{pr}_{t;R[s]} \circ I_{t,s;R}: R[t,s] \to R[s]/[t], \quad J_{t,s;R}^N: \tau_{s;R[t^{-1}]}^{\leq N} \circ \text{pr}_{t;R[s]} \circ I_{t,s;R}: R[t,s] \to t^{-1}R[t^{-1}][s].$$

We can check easily that the image of $J_{t,s;R}^*$ (resp. $J_{t,s;R}^N$) is included in $R[t,s]$ (resp. $t^{-1}R[t^{-1},s]$).

For simplicity, we omit the ring $R$ from our notations. The following Lemma 2.15 and Lemma 2.18 are fundamental for the proofs of Theorem 2.5 and Theorem 2.10.
Lemma 2.15. Let \( n \) be a positive integer. Then we have the following identities:

\[
I_t(t^n) = \frac{t^n}{n},
\]
\[
I_t((1-t)^n - 1) = \sum_{j=1}^{n} \frac{(1-t)^j - 1}{j},
\]
\[
J^*_t,s(t^n) = \sum_{j=1}^{n} \frac{t^{n-j}s^j}{j},
\]
\[
J^*_t,s((1-t)^n - 1) = \sum_{j=1}^{n} \frac{(1-t)^{n-j}\{(1-s)^j - 1\}}{j}.
\]

Proof. The equality (11) can be easily checked. We show the equality (12). Set \( T := 1-t \). Then the left hand side of (12) equals to

\[-\int \frac{T^k - 1}{1 - T} dT = \int \sum_{j=0}^{k-1} T^j dT = \sum_{j=1}^{k} \frac{T^j - 1}{j}.\]

By the definition of \( T \), we obtain the equality (12). Let us show the equality (13). By the equality (10), the following equalities hold:

\[
J^*_t,s(t^n) = \text{pr}_{t,R[s]}(t^n I_{t,s;R}(1)) = \text{pr}_{t,R[s]} \left( \sum_{j=1}^{\infty} s^j t^{n-j} \right) = \sum_{j=1}^{n} s^j t^{n-j}.\]

Finally, we show the equality (14). Note that the following equalities hold:

\[
\frac{(1-t)^n}{t-s} = -\frac{(1-t)^{n-1}}{1 - \frac{s-1}{t-1}} = -(1-t)^{n-1} \left( \frac{1 - \left(\frac{s-1}{t-1}\right)^n}{1 - \frac{s-1}{t-1}} + \frac{\left(\frac{s-1}{t-1}\right)^n}{1 - \frac{s-1}{t-1}} \right)
\]

\[= -(1-t)^{n-1} \sum_{j=0}^{n-1} \left( \frac{s-1}{t-1} \right)^j + \frac{(1-s)^n}{t-s}
\]

\[= -\sum_{j=0}^{n-1} (1-s)^j(1-t)^{n-j-1} + \frac{(1-s)^n}{t-s}.
\]

As \( J^*_t,s((1-t)^n - 1) = J^*_t,s((1-t)^n) = \int \left( -\sum_{j=0}^{n-1} (1-s)^j(1-t)^{n-j-1} \right) ds = \sum_{j=1}^{n} \frac{(1-t)^{n-j}\{(1-s)^j - 1\}}{j}.\)

This completes the proof of the lemma.

Before we give the lemma for \( J^*_t,s \), we prepare the following two auxiliary lemmas:
Lemma 2.16 (cf. [21, Proof of Lemma 4.1]). Let $N$ be a positive integer. We have the following polynomial identities in $R[t]$

\begin{align}
\sum_{n=1}^{N} (-1)^n \binom{N}{n} \frac{t^n}{n} &= \sum_{n=1}^{N} \frac{(1-t)^n - 1}{n}, \\
\sum_{n=1}^{N} \frac{t^n}{n} &= \sum_{n=1}^{N} (-1)^n \binom{N}{n} \frac{(1-t)^n - 1}{n}.
\end{align}

(15) 

(16)

\begin{proof}
First, we remark that $(1-t)^N - 1 = \sum_{n=1}^{N} \binom{N}{n} (-t)^n$. Then by applying $I_t$ to both sides and using Lemma 2.15 (11) and (12), we obtain the identity (15). The identity (16) is obtained by the substitution $t \mapsto 1 - t$ and the Euler’s identity (3), which is a special case of the identity (15).
\end{proof}

Lemma 2.17. Let $j$ and $n$ be non-negative integers satisfying $j \leq n$. Then we have the following polynomial identity in $R[t]$

\[
\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} t^k = \binom{n}{j} t^j (1 + t)^{n-j}.
\]

Proof. By the binomial expansion, we have 

\[
(t + s + ts)^n = (t + (1 + t)s)^n = \sum_{j=0}^{n} \binom{n}{j} t^j (1 + t)^{n-j} s^{n-j}.
\]

On the other hand, we have 

\[
(t + s + ts)^n = \{t(1+s) + s\}^n = \sum_{k=0}^{n} \binom{n}{k} t^k (1+s)^k s^{n-k} \]

\[
= \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} t^j (1+s)^j s^{n-k-j} \right) s^{n-j}.
\]

Comparing the coefficients of $s^{n-j}$.

\begin{proof}
By the binomial expansion, we have 

\[
\sum_{k=0}^{n} \binom{n}{k} t^k \sum_{j=0}^{k} \binom{k}{j} s^{k-j} s^{n-k} = \sum_{j=0}^{n} \left( \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} t^k \right) s^{n-j}.
\]

\end{proof}

Lemma 2.18. Let $N$ and $n$ be positive integers. Then we have the following identities in $R[t^{\pm 1}, s]$

\begin{align}
J_{t,s}^N(t^n) &= \sum_{j=n+1}^{N} \frac{s^j t^{n-j}}{j}, \\
J_{t,s}^N((1-t)^n - 1) &= -\sum_{j=1}^{n} \frac{(1-t)^n \{(1-s)^j - 1\}}{j} + \left( \sum_{j=1}^{N} \frac{(s/t)^j}{j} \right) \{(1-t)^n - 1\}.
\end{align}

(17) 

(18)

Here, we understand the summation in the right hand side of the equality (17) as 0 if $n+1$ is greater than $N$. 

Proof. The equality (17) is an immediate consequence of the equality (10). We show the equality (18). By the equality

\[ I_{t,s;R}((1-t)^n) = (1-t)^n \sum_{j=1}^{\infty} \frac{s^j t^{-j}}{j} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \sum_{j=1}^{\infty} \frac{s^j t^{k-j}}{j} \right), \]

we have

\[ J^N_{t,s}((1-t)^n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \sum_{j=k+1}^{N} \frac{s^j t^{k-j}}{j} \right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \sum_{j=1}^{N} \frac{s^j t^{k-j}}{j} - \sum_{j=1}^{k} \frac{s^j t^{k-j}}{j} \right) \]

\[ = \sum_{j=1}^{N} \frac{(s/t)^j}{j} (1-t)^n - \sum_{n \geq k \geq j \geq 1} (-1)^k \binom{n}{k} \frac{s^j t^{k-j}}{j}. \]

(19)

Since the equality

\[ \sum_{n \geq k \geq j \geq 1} (-1)^j \binom{j}{k} \left( \frac{(1-s/t)^j}{j} - 1 \right) \]

holds by Lemma 2.16 (16), we have

\[ \sum_{n \geq k \geq j \geq 1} (-1)^j \binom{j}{k} \left( \frac{(1-s/t)^j}{j} - 1 \right) = \sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j}{j} - \frac{1}{j} t^k. \]

Furthermore, by Lemma 2.17, we have

\[ \sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j}{j} t^k = (1-t)^n \sum_{j=1}^{n} \binom{n}{j} \frac{1}{j} \left\{ (1-s/t)^j \right\} - (1-t)^n \sum_{j=1}^{n} \binom{n}{j} \frac{1}{j} \left\{ (1-s/t)^j \right\} \]

Therefore, according to Lemma 2.16 (15), we can delete the binomial coefficients completely as follows:

\[ \sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j}{j} - \frac{1}{j} t^k = (1-t)^n \sum_{j=1}^{n} \frac{1}{j} \left\{ (1-s/t)^j \right\} - (1-t)^n \sum_{j=1}^{n} \frac{1}{j} \left\{ (1-s/t)^j \right\} \]

\[ = (1-t)^n \sum_{j=1}^{n} \frac{1}{j} \left\{ (1-s/t)^j \right\} - (1-t)^n \sum_{j=1}^{n} \frac{1}{j} \left\{ (1-s/t)^j \right\} \]

\[ = \sum_{j=1}^{n} \frac{(1-t)^n}{j} \left\{ (1-s/t)^j - 1 \right\}. \]

Hence, we have the desired identity by the equalities (19) and \( J^N_{t,s}(1) = \sum_{j=1}^{N} \frac{(s/t)^j}{j}. \)
2.4. Proofs of Theorem 2.5 and Theorem 2.10

Proof of Theorem 2.5. We show this theorem by the induction on the weight $w$ of the index. If $w = 1$, the assertion of the theorem is nothing but Lemma 2.16. We show only the equality (4) because the proof of the equality (5) is completely the same. Now, we assume that the assertion holds for an index $k = (k_1, \ldots, k_m)$. Then it is sufficient to show that the assertions also hold for the indices $k_1 := (k_1, \ldots, k_m + 1)$ and $k_2 := (k_1, \ldots, k_m, 1)$.

First, we consider the case $k_1$. By Lemma 2.15 (11), we have

$$I_{tm} \left( \sum' (-1)^{n_1} \left( \frac{N}{n_1} \right) t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m} \right) = \sum' (-1)^{n_1} \left( \frac{N}{n_1} \right) t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m} n_1^{k_1} \cdots n_m^{k_m+1},$$

where the summation $\sum'$ runs over $N \geq n_1 \geq \cdots \geq n_m \geq 1$ and $I_{tm} := I_{tm; R[t_1, \ldots, t_m-1]}$. On the other hand, by Lemma 2.15 (12), we have

$$= \sum_{N \geq n_1 \geq \cdots \geq n_{w+1} \geq 1} \frac{(1-t_1)^{n_1-n_1+1} \cdots (1-t_{m-1})^{n_{m-1}-n_{m-1}+1} \{(1-t_m)^{n_m} - 1\}}{n_1 \cdots n_w},$$

where $l_1 = k_1, l_2 = k_1 + k_2, \ldots, l_m = k_1 + \cdots + k_m (= w)$. Thus, the equality (4) in the theorem also holds for $k_1$ by the induction hypothesis.

Next, we check the equality for the index $k_2$. By Lemma 2.15 (13), we have

$$J^*_{tm, t_{m+1}} \left( \sum' (-1)^{n_1} \left( \frac{N}{n_1} \right) t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m} \right) = \sum'' (-1)^{n_1} \left( \frac{N}{n_1} \right) t_1^{n_1-n_2} \cdots t_m^{n_m-n_{m+1}} t_{m+1}^{n_{m+1}},$$

where the summation $\sum''$ runs over $N \geq n_1 \geq \cdots \geq n_{m+1} \geq 1$ and $J^*_{tm, t_{m+1}} := J^*_{tm, t_{m+1}; R[t_1, \ldots, t_{m-1}]}$. On the other hand, by Lemma 2.15 (14), we have

$$= \sum_{N \geq n_1 \geq \cdots \geq n_{w+1} \geq 1} \frac{(1-t_1)^{n_1-n_1+1} \cdots (1-t_{m-1})^{n_{m-1}-n_{m-1}+1} \{(1-t_m)^{n_m} - 1\}}{n_1 \cdots n_w}.$$

Using the induction hypothesis, the assertion of the equality (4) holds for the index $k_2$. This completes the proof of Theorem 2.5.

Proof of Theorem 2.10. We show this theorem by the induction on the weight $w$ of the index. If $w = 1$, the assertion of the theorem is nothing but Lemma 2.16. We show only the equality (7) because the proof of the equality (8) is completely the same. Now, we assume that the assertion holds for an index $k = (k_1, \ldots, k_m)$. Then it is sufficient to show that the assertions also hold for the indices $k_1 := (k_1 + 1, \ldots, k_m)$ and $k_2 := (1, k_1, \ldots, k_m)$.

First, we consider the case $k_1$. By Lemma 2.15 (11), we have

$$I_{t_1} \left( \sum (-1)^{n_m} \left( \frac{N}{n_m} \right) t_1/t_2 \cdots t_{m-1}/t_m \right) = \sum' (-1)^{n_1} \left( \frac{N}{n_1} \right) (t_1/t_2)^{n_1} \cdots (t_{m-1}/t_m)^{n_m} n_1^{k_1} \cdots n_m^{k_m+1},$$

which completes the proof of Theorem 2.10.
where the summation \( \sum' \) runs over \( N + 1 > n_1 > \cdots > n_m > 0 \) and \( I_t^1 := I_{t_1; R[t_2^{+1}, \ldots, t_m^{+1}]} \). On the other hand, by Lemma 2.15 (12), we have

\[
I_{t_1}(R. H. S. \text{ of the equality (7)}) = \\
(-1)^{m-1} \sum_{N \geq n_1 \geq \cdots \geq n_{w+1} \geq 1} (1 - t_m)^{n_1 - n_1 + 1} \cdots (1 - t_2)^{n_{m-1} - n_{m-1} + 1} \left\{ (1 - t_1)^{n_{m+1} - 1} \right\} \\
+ \sum_{j=1}^{m-1} (-1)^{m-j-1} \left( \sum_{N+1 > n_1 > \cdots > n_j > 0} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1+1} \cdots n_j^{k_j}} \right) \\
\times \left( \sum_{N \geq n_1 \geq \cdots \geq n_{m-j} \geq 1} (1 - t_m)^{n_1 - n_1 + 1} \cdots (1 - t_{j+2})^{n_{m-j-1} - n_{m-j-1} + 1} \left\{ (1 - t_{j+1})^{n_{m-j} - 1} \right\} \\
\right)
\]

where \( l_1 = k_m, l_2 = k_m + k_{m-1}, \ldots, l_m = k_m + \cdots + k_1 (= w) \). Thus, the equality (7) in the theorem also holds for the index \( k_1 \) by the induction hypothesis.

Next, we check the equality for the index \( k_2 \). By Lemma 2.18 (17), we have

\[
J_{t_1, t_0}^N \left( \sum' (-1)^m \frac{N}{n_1 n_2 \cdots n_{m-1} n_m} \right) = \sum'' (-1)^m \left( \frac{N}{n_1 n_2 \cdots n_{m-1} n_m} \right) \left( \sum \frac{(t_1/t_2)^{n_1} \cdots (t_m/t_{m-1})^{n_m}}{n_0 n_1^{k_1} \cdots n_m^{k_m}} \right)
\]

where the summation \( \sum'' \) runs over \( N + 1 > n_0 > n_1 > \cdots > n_m > 0 \) and \( J_{t_1, t_0}^N := J_{t_1, t_0; R[t_2^{+1}, \ldots, t_m^{+1}]} \). On the other hand, by Lemma 2.18 (18), we have

\[
(-1)^m J_{t_1, t_0}^N (R. H. S. \text{ of the equality (7)}) = \\
\left( \sum_{n_0 = 1}^{N} \frac{(t_0/t_1)^{n_0}}{n_0} \right) \left( \sum_{N \geq n_1 \geq \cdots \geq n_u \geq 1} (1 - t_m)^{n_1} \cdots (1 - t_2)^{n_{m-1}} \left\{ (1 - t_1)^{n_{m+1} - 1} \right\} \\
+ \sum_{j=1}^{m-1} (-1)^{j-1} \left( \sum_{N+1 > n_0 > n_1 > \cdots > n_j > 0} \frac{(t_0/t_1)^{n_0} \cdots (t_j/t_{j+1})^{n_j}}{n_0 n_1^{k_1} \cdots n_j^{k_j}} \right) \times \left( \sum_{N \geq n_1 \geq \cdots \geq n_{m-j} \geq 1} (1 - t_m)^{n_1} \cdots (1 - t_{j+2})^{n_{m-j-1}} \left\{ (1 - t_{j+1})^{n_{m-j} - 1} \right\} \\
\right)
\]

Using the induction hypothesis, the assertion of the equality (7) holds for the index \( k_2 \). This completes the proof of Theorem 2.10. \( \square \)

3. Functional equations of finite multiple polylogarithms

3.1. The ring \( \mathcal{A}_{n,R} \)

Kaneko and Zagier defined finite multiple zeta(-star) values as elements of the \( Q \)-algebra \( \mathcal{A} = (\prod_p \mathbb{F}_p) \otimes \mathbb{Z} \) where \( p \) runs over the set of all rational primes. See [8] and their forthcoming paper [9]. The ring \( \mathcal{A} \) has been used in a different context by Kontsevich [13, 2.2]. Now, we introduce more general rings of such a type for the definition of finite multiple polylogarithms.
Definition 3.1. Let $R$ be a commutative ring and $\Sigma$ a family of ideals of $R$. Then we define the ring $\mathcal{A}_{n,R}^\Sigma$ for each positive integer $n$ as follows:

$$\mathcal{A}_{n,R}^\Sigma := \left( \prod_{I \in \Sigma} R/I^n \right) / \left( \bigoplus_{I \in \Sigma} R/I^n \right).$$

Since we use only the case $\Sigma = \{ pR \mid p$ is a rational prime $\}$, we omit the notation $\Sigma$ in the rest of this paper. We denote $\mathcal{A}_{n,Z} = \mathcal{A}_n$ and $\mathcal{A}_{1,R} = \mathcal{A}_R$. Then the ring $\mathcal{A}_{1,Z}$ coincides with $\mathcal{A}$. We will define the finite multiple polylogarithms as elements of $\mathcal{A}_{n,Z[t_1, \ldots, t_m]}$ (that is not equal to the polynomial ring $\mathcal{A}_n[t_1, \ldots, t_m]$). Note that $\mathcal{A}_{n,R}$ is a $\mathbb{Q}$-algebra in some important case even if $R$ is not a $\mathbb{Q}$-algebra.

Example 3.2. We often use

$$\mathcal{A}_{Z[t]} = \left( \prod_p \mathbb{F}_p[t] \right) / \left( \bigoplus_p \mathbb{F}_p[t] \right)$$

in the next section. Here, the direct product and the direct sum run over all rational primes and $\mathcal{A}_{Z[t]}$ coincides with $\mathcal{B}$ defined by Ono and Yamamoto in their paper [18].

We denote each element of $\mathcal{A}_{n,R}$ as $(a_p)_p$ where $a_p \in R/p^n R$, so $(a_p)_p = (b_p)_p$ holds if and only if $a_p = b_p$ for all but finitely many rational primes $p$. We may use the notation $(a_p)_p \in \mathcal{A}_{n,R}$ even if $a_p$ is not defined for finitely many rational primes $p$. See Example 3.3 (3). We define the element $p_n \in \mathcal{A}_n$ to be $(p \mod p^n)_p$ and we denote it by $p$ when $n$ is clear from the context. Note that $\mathcal{A}_n/p^m \mathcal{A}_n = \mathcal{A}_m$ for $n \geq m$.

Example 3.3. We give some typical examples of elements of $\mathcal{A}_{n,R}$.

1. $t^p := (t^p)_p \in \mathcal{A}_{n,Z[t]}$.

2. For any rational number $r$, we define $r^p$ (resp. $r$) to be $(r^p)_p$ (resp. $(r)_p$) in $\mathcal{A}$. Then $r^p = r \in \mathcal{A}$ holds by Fermat’s little theorem.

3. Let $k$ be a positive integer greater than 2. We use the notation $B_{p-k}$ as $B_{p-k} = (B_{p-k} \mod p^n)_p \in \mathcal{A}_n$,

where $B_m$ is the $m$-th Bernoulli number defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_m \frac{t^m}{m!}.$$ 

By the von Staudt-Clausen theorem [26, Theorem 5.10], $B_{p-k}$ is a $p$-adic integer for a rational prime greater than $k$. Therefore, the element $B_{p-k}$ is well-defined as an element of $\mathcal{A}_n$ and conjecturally, it is non-zero by the conjecture that there are infinitely many regular primes.
The following lemma will be used for deducing the functional equations of the finite multiple polylogarithms from the generalizations of Euler’s identity obtained in Section 2.

**Lemma 3.4.** Let $n$ be a positive integer. Then we have

$$(-1)^n \left( \binom{p-1}{n} \right) = 1 - pH_n \text{ in } \mathcal{A}_2,$$

where $H_n = \sum_{j=1}^{n} 1/j$ is the $n$-th harmonic number.

Let $p$ be an odd prime number greater than $n$. Then the following congruence is satisfied:

$$H_{p-n-1} \equiv H_n \pmod{p}.$$ (21)

Here, we define $H_0$ to be 1.

**Proof.** For each prime number $p > n$, we have the following congruence:

$$(-1)^n \left( \binom{p-1}{n} \right) = \prod_{j=1}^{n} \left( 1 - \frac{p}{j} \right) \equiv 1 - pH_n \pmod{p^2}.$$ (22)

Therefore the first assertion holds. Let $p > n + 1$. By the substitution $n \mapsto p - n$, we have

$$H_{p-n-1} = \sum_{k=1}^{p-n-1} \frac{1}{k} = \sum_{k=n+1}^{p-1} \frac{1}{p-k} \equiv - \sum_{k=n+1}^{p-1} \frac{1}{k} = -(H_{p-1} - H_n) \equiv H_n \pmod{p}.$$ (23)


### 3.2. Definitions and functional equations of finite multiple polylogarithms

First, we recall the definition of the finite multiple zeta(-star) values (FMZ(S)Vs).

**Definition 3.5.** Let $n$ be a positive integer and $\mathbf{k} = (k_1, \ldots, k_m)$ an index. Then we define the **finite multiple zeta value** $\zeta_{\mathcal{A}_n}(\mathbf{k})$ by

$$\zeta_{\mathcal{A}_n}(\mathbf{k}) := \left( \sum_{p>n_1>\cdots>n_m>0} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \pmod{p^n} \right)_p \in \mathcal{A}_n$$

and we define the **finite multiple zeta-star value** $\zeta^\star_{\mathcal{A}_n}(\mathbf{k})$ by

$$\zeta^\star_{\mathcal{A}_n}(\mathbf{k}) := \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \pmod{p^n} \right)_p \in \mathcal{A}_n.$$

**Remark 3.6.** Several people use the notation $\zeta_{\mathcal{A}_n}(\mathbf{k})$ instead of $\zeta^\star_{\mathcal{A}_n}(\mathbf{k})$ for $\bullet \in \{\emptyset, \star\}$. Therefore we have to be careful when we read other papers.

For later use, we summarize known results about FMZ(S)Vs.
Proposition 3.7. Let \( m, k, k_1, k_2, k_3 \) be positive integers, \( k = (k_1, \ldots, k_m) \) an index and \( \bullet \in \{\emptyset, \ast\} \). Then the following equalities hold:

\[
\zeta_{A_2}^*(\{k\}^m) = (-1)^{m-1} \zeta_{A_2}^* \{k\}^m = (-1)^{m-1} k \frac{B_{p^m-k-1}}{mk+1} \quad (23)
\]

\[
\zeta_{A_2}^*(k_1, k_2) = (-1)^{k_1} \left( \frac{w+1}{k_1} \right) - (-1)^{k_1} k_2 \left( \frac{w+1}{k_1} \right) - w \right \) \frac{B_{p-w^{-1}}}{w+1} \quad (25)
\]

if \( w = k_1 + k_2 \) is even,

\[
\zeta_{A_2}^*(k_1, k_2, k_3) = -\zeta_{A_2}^*(k_1, k_2, k_3) = \frac{1}{2} \left\{ (-1)^{k_3} \left( \frac{w'}{k_3} \right) - (-1)^{k_1} \left( \frac{w'}{k_3} \right) \right \} \frac{B_{p-w'}}{w'} \quad (27)
\]

if \( w' = k_1 + k_2 + k_3 \) is odd.

Proof. The equalities (23), (24), (25), (26), and (27) are [28, Theorem 1.6, [28, Theorem 3.1] (cf. [6, Theorem 6.1]), [28, Theorem 3.2], [28, Theorem 3.2], and [28, Theorem 3.5] (cf. [6, Theorem 6.2]), respectively. The equalities (28), (29), and (30) are [6, Theorem 4.5], [6, Theorem 4.6], and [6, Theorem 5.1], respectively.

Now, we define the finite multiple polylogarithms which are main objects in this paper.

Definition 3.8. Let \( n \) be a positive integer and \( k = (k_1, \ldots, k_m) \) an index. Then we define the various multi-variable finite multiple polylogarithms as follows:

The finite harmonic multiple polylogarithm (FHMP):

\[
\mathcal{L}_{A_n,k}^*(t_1, \ldots, t_m) := \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)^n \in \mathcal{A}_{n,\mathbb{Z}[t_1, \ldots, t_m]}.
\]

The finite harmonic star-multiple polylogarithm (FHSMP):

\[
\mathcal{L}_{A_n,k}^{**}(t_1, \ldots, t_m) := \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right)^n \in \mathcal{A}_{n,\mathbb{Z}[t_1, \ldots, t_m]}.
\]
The finite shuffle multiple polylogarithm (FSMP):

\[ \mathcal{L}_m^{\bullet}(t_1, \ldots, t_m) := \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1-1} \cdots t_m^{n_m-1} \mod p^n}{n_1 \cdots n_m} \right) \in \mathcal{A}_{n, Z}[t_1, \ldots, t_m]. \]

The finite shuffle star-multiple polylogarithm (FSSMP):

\[ \mathcal{L}_m^{\bullet, \star}(t_1, \ldots, t_m) := \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{t_1^{n_1-1} \cdots t_m^{n_m-1} \mod p^n}{n_1 \cdots n_m} \right) \in \mathcal{A}_{n, Z}[t_1, \ldots, t_m]. \]

We also define 1-variable finite (star-)multiple polylogarithms (F(S)MP) as follows:

\[ \mathcal{L}_{A, n, k}^\bullet(t) := \mathcal{L}_{A, n, k}^\bullet(t, \{1\}^{m-1}) = \mathcal{L}_{A, n, k}^\bullet(\{1\}^{m-1}, t) \in \mathcal{A}_{n, Z}[t], \]

\[ \mathcal{L}_{A, n, k}^{\star}(t) := \mathcal{L}_{A, n, k}^{\star}(\{1\}^{m-1}, t) = \mathcal{L}_{A, n, k}^{\star}(\{1\}^{m-1}, t) \in \mathcal{A}_{n, Z}[t], \]

where \( \bullet \in \{0, \star\} \). We call \( \mathcal{L}_{A, n, m}(t) = \mathcal{L}_{A, n, m}^\bullet(t) \) the \( m \)-th finite polylogarithm.

**Remark 3.9.** The original definition of the \( m \)-th finite polylogarithm by Elbaz-Vincent and Gangl is the \( p \)-component of \( \mathcal{L}_{A, n, m}(t) \) in \( \mathbb{F}_p[t] \) (cf. [2, Definition 5.1]).

**Remark 3.10.** Let \( R \) be a commutative ring. For any subset \( \{i_1, \ldots, i_h\} \) of \( \{1, \ldots, m\} \) and \( r_1, \ldots, r_h \in R \), the substitution mapping

\[ \mathcal{A}_{n, Z}[t_1, \ldots, t_m] \rightarrow \mathcal{A}_{n, R[t_{j_1}, \ldots, t_{j_h}]} \]

defined by

\[ (f_p(t_1, \ldots, t_m))_p \mapsto (f_p(t_1, \ldots, t_m)|_{t_1 = r_1, \ldots, t_h = r_h})_p \]

where \( \{j_1, \ldots, j_h\} \) is the complement of \( \{i_1, \ldots, i_h\} \) with respect to \( \{1, \ldots, m\} \). For example, we have

\[ \mathcal{L}_{A, n, k}^\bullet(1) = \mathcal{L}_{A, n, k}^{\star}(1) = \zeta_{A, n}(k) \in \mathcal{A}_n \]

for \( \bullet \in \{0, \star\} \). Our definition of FMPs is natural in this sense.

The following proposition is a generalization of Proposition 3.7 (28) (cf. [25, Lemma 5.4]):

**Proposition 3.11** (Reversal formulas). Let \( \mathbb{K} = (k_1, \ldots, k_m) \) be an index and \( \bullet \in \{0, \star\} \).

Then we have the following equality in \( \mathcal{A}_{2, Z[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]} \):

\[ \mathcal{L}_{A, 2, k}^\bullet(t_1, \ldots, t_m) = (-1)^{w_2(k)}(t_1 \cdots t_m)^{\mathbb{K}} \mathcal{L}_{A, 2, \mathbb{K}}^\bullet(t_m^{\pm 1}, \ldots, t_1^{\pm 1}) + p \sum_{i=1}^{m} k_i \mathcal{L}_{A, 2, k_i \mathbb{K}}^\bullet(t_m^{\pm 1}, \ldots, t_1^{\pm 1}). \] (31)

In particular, we have

\[ \mathcal{L}_{A, 2, k}^\bullet(t_1, \ldots, t_m) = (-1)^{w_2(k)}(t_1 \cdots t_m)^{\mathbb{K}} \mathcal{L}_{A, 2, \mathbb{K}}^\bullet(t_m^{\pm 1}, \ldots, t_1^{\pm 1}), \] (32)

\[ \mathcal{L}_{A, 2, k}^\bullet(t) = (-1)^{w_2(k)}p \mathcal{L}_{A, 2, k}^\bullet(t_1), \quad \mathcal{L}_{A, 2, k}^\bullet(t) = (-1)^{w_2(k)}p \mathcal{L}_{A, 2, k}^\bullet(t_1). \] (33)
Proof. We show only the case $\bullet = \emptyset$ since the proof for the case $\bullet = \star$ is similar. By using the substitution trick $n_i \mapsto p - n_i$, we have

$$
\mathcal{L}_{A_2,k}^*(t_1, \ldots, t_m) = \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p^2 \right)_p
$$

$$
= \left( \sum_{p > n_1 > \cdots > p - n_m > 0} \frac{p^{p-n_1} \cdots p^{p-n_m} \mod p^2}{(p - n_1)^{k_1} \cdots (p - n_m)^{k_m}} \right)_p
$$

$$
= (-1)^{\text{wt}(k)}(t_1 \cdots t_m) \mathcal{P} \left( \sum_{p > n_1 > \cdots > n_1 > 0} \frac{(p + n_m)^{k_m} \cdots (p + n_1)^{k_1} t_1^{-n_1} \cdots t_m^{-n_1}}{n_1^{2k_1} \cdots n_m^{2k_m}} \mod p^2 \right)_p.
$$

Since $(p + n_m)^{k_m} \cdots (p + n_1)^{k_1} = n_m^{k_m} \cdots n_1^{k_1} + p \sum_{i=1}^{m} k_i n_i^{k_m} \cdots n_i^{-k_1} \cdot n_i^{k_1} \mod p^2$, we have the equality (31).

Our main results in this paper are Theorem 3.12, Corollary 3.13 and Theorem 3.15 below:

**Theorem 3.12.** Let $k = (k_1, \ldots, k_m)$ be an index of weight $w$. Then we have the following functional equation in $A_{2,\mathbb{Z}[1,\ldots,t_m]}$ for FSSMPs:

$$
\mathcal{L}_{A_2,k}^m(t_1, \ldots, t_m) + \left( \mathcal{L}_{A_2,\{1\},k}^m(1, t_1, \ldots, t_m) - \mathcal{L}_{A_2,\{1\} \oplus k}^m(t_1, \ldots, t_m) \right) \mathcal{P}
$$

$$
= \mathcal{L}_{A_2,\{1\}}^m \left( \{1\}^{k_1-1, 1 - t_1, 1 - t_1, 1 - t_2, \ldots, 1}^{k_m-1} \right) \mathcal{P}
$$

$$
- \mathcal{L}_{A_2,\{1\}}^m \mathcal{P} \left( \{1\}^{k_1-1} - 1 - t_1, \{1\}^{k_m-1, 1 - t_1, 1 - t_m-1, 1}^{k_m} \right). \quad (34)
$$

Proof. By Lemma 3.4 (20), we have

$$
\mathcal{L}_{A_2,k}^m(t_1, \ldots, t_m) \mathcal{P} \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p^2 \right)_p
$$

$$
= \mathcal{L}_{A_2,k}^m(t_1, \ldots, t_m) \mathcal{P} \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{t_1^{n_1-1} \cdots t_m^{n_m-1}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p^2 \right)_p
$$

$$
= \mathcal{L}_{A_2,k}^m(t_1, \ldots, t_m) \mathcal{P} \left( \sum_{p > n_1 > \cdots > n_m > 0} \frac{H_{n_1} t_1^{n_1-1} \cdots t_m^{n_m-1}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p^2 \right)_p.
$$
By substitutions \( n_i \mapsto p - n_i \) and Lemma 3.4 (21), we have

\[
\left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{H_{n_1} t_1^{n_1-n_2} \cdots t_{m-1}^{n_{m-1}-n_m} t_m^{p-n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right) 
\]

\[
= \left( \sum_{p-1 \geq p-n_1 \geq \cdots \geq p-n_m \geq 1} \frac{H_{p-n_1} t_1^{(p-n_1)-(p-n_2)} \cdots t_{m-1}^{(p-n_{m-1})-(p-n_m)} t_m^{p-n_m}}{(p-n_1)^{k_1} \cdots (p-n_m)^{k_m}} \mod p \right) 
\]

\[
= (-1)^{\text{wt}(k)} \left( \sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{t_1^{n_2-1} \cdots t_{m-1}^{n_{m-1}-1} t_m^{p-n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \mod p \right) 
\]

\[
= - \frac{\sum_{p-1 \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_0 n_1^{k_1} \cdots n_m^{k_m}}}{n_1^{k_1+1} \cdots n_m^{k_m}} \mod p 
\]

\[
= - \mathbb{F}_{A_2, \{1\} \cup k_1}^{\text{ind}} (t_1, \ldots, t_m) + \mathbb{F}_{A_2, \{1\} \cup k_1}^{\text{ind}} (t_1, \ldots, t_m) 
\]

Therefore, we have the desired functional equation by Theorem 2.5.

When we substitute 1 for some of the variables \( t_1, \ldots, t_{m-1} \) in (34), we have the following functional equation:

**Corollary 3.13.** Let \( \mathbb{F} = (k_1, \ldots, k_m) \) be an index, \( S = \{i_1, \ldots, i_h\} \) a subset of \( \{1, \ldots, m-1\} \), and \( \{j_1, \ldots, j_h\} \) the complement of \( \{i_1, \ldots, i_h\} \) with respect to \( \{1, \ldots, m-1\} \). We define \( \mathbb{F}_S := (k_1 + \cdots + k_{i_1}, k_{i_1+1} + \cdots + k_{i_2}, \ldots, k_{i_h+1} + \cdots + k_m) \) and \( m' := \text{dep}(\mathbb{F}_S) \).

Then we have the following functional equation in \( A_2 \otimes [t_1, \ldots, t_{m-1}] \) for FSSMPs:

\[
\mathbb{F}_{A_2, \mathbb{F}_S} (\{1\}^{l_1}, 1 - t_1, \ldots, t_m) = \mathbb{F}_{A_2, \mathbb{F}_S} (\{1\}^{l_1}, 1 - t_1, \ldots, t_m) - \mathbb{F}_{A_2, \mathbb{F}_S} (\{1\}^{l_1}, 1 - t_1, \ldots, t_m) \mod p 
\]

(35)

where \( l_1 = k_1 + \cdots + k_{j_1} - j_1, l_2 = k_{j_1+1} + \cdots + k_{j_2} - j_2 + j_1, \ldots, l_{h'} = k_{j_{h'-1}+1} + \cdots + k_{j_{h'}} - j_{h'} + j_{h'}-1, \) and \( M' = k_1 + \cdots + k_{j_{h'}} - j_{h'} + h' \).
Proof. This is obtained by combining the proof of Theorem 3.12 and Corollary 2.6.

Remark 3.14. In particular, we have the following functional equation in $\mathcal{A}_{2, \mathbb{Z}[t]}$ (cf. Corollary 2.7):

$$
\overline{L}_{A_2, \mathbb{k}}^*(t) + (\overline{L}_{A_2, \mathbb{n}}^*(t) - \overline{L}_{A_2, e_1 \oplus \mathbb{k}}^*(t))p = \overline{L}_{A_2, \mathbb{k}^\vee}^*(1-t) - \zeta_{A_2}^*(\mathbb{k}^\vee).
$$

(36)

Therefore, we also have the functional equation (1) in Introduction. The case $t = 1$ gives the Hoffman duality (Proposition 3.7 (29)) and its generalization in $\mathcal{A}_2$ ([28, Theorem 2.11]):

$$
\zeta_{A_2}^*(\mathbb{k}) + (\zeta_{A_2}^*(\{1\} \sqcup \mathbb{k}) - \zeta_{A_2}^*(e_1 \oplus \mathbb{k}))p = -\zeta_{A_2}^*(\mathbb{k}^\vee).
$$

(37)

Theorem 3.15. Let $n$ be a positive integer and $\mathbb{k} = (k_1, \ldots, k_m)$ an index. Then we have the following functional equation in $\mathcal{A}_{n, \mathbb{Z}[t_1, \ldots, t_m]}$:

$$
\sum_{j=0}^{m} (-1)^j \mathcal{L}_{A_n,(k_1, \ldots, k_j)}^*(t_1, \ldots, t_j) \mathcal{L}_{A_n,(k_m, \ldots, k_{j+1})}^*(t_m, \ldots, t_{j+1}) = 0.
$$

(38)

Here, we consider $\mathcal{L}_{A_n,(k_1, \ldots, k_j)}^*(t_1, \ldots, t_j)$ (resp. $\mathcal{L}_{A_n,(k_m, \ldots, k_{j+1})}^*(t_m, \ldots, t_{j+1})$) as 1 when $j = 0$ (resp. $j = m$).

Proof. This is an immediate consequence of Theorem 2.11.

Corollary 3.16. Let $n$, $k$, and $m$ be positive integers and $\mathbb{k} = (k_1, \ldots, k_m)$ an index. Then the following equalities hold:

$$
\mathcal{L}_{A_n,(k,m)}^*(\{1\}^{i-1}, t, \{1\}^{m-i}) + (-1)^m \mathcal{L}_{A_n,(k,m)}^*(\{1\}^{m-i}, t, \{1\}^{i-1}) = 0,
$$

(39)

$$
(-1)^{m-1} \mathcal{L}_{A_n,(k)}^*(t) = \mathcal{L}_{A_n,(k)}^*(t) + \sum_{j=1}^{m-1} (-1)^j \mathcal{L}_{A_n,(k_1, \ldots, k_j)}^*(t) \zeta_{A_n}^*(k_m, \ldots, k_{j+1}),
$$

(40)

$$
(-1)^{m-1} \mathcal{L}_{A_n,(k)}^*(t) = \mathcal{L}_{A_n,(k)}^*(t) + \sum_{j=1}^{m-1} (-1)^j \zeta_{A_n}^*(k_1, \ldots, k_j) \mathcal{L}_{A_n,(k_m, \ldots, k_{j+1})}^*(t),
$$

(41)

$$
\sum_{j=0}^{m} (-1)^j \zeta_{A_n}^*(k_1, \ldots, k_j) \zeta_{A_n}^*(k_m, \ldots, k_{j+1}) = 0,
$$

(42)

Here, we consider $\zeta_{A_n}^*(\emptyset)$ as 1 for $\bullet \in \{\emptyset, *\}$.

Proof. We obtain the equality (39) by the substitution $t_1 = \cdots = t_{i-1} = t_{i+1} = \cdots = t_m = 1$, $t_i = t$ and Lemma 3.7 (23). The equalities (40), (41), and (42) are clear.

Remark 3.17. The equality (39) has been proved by Tauraso and J. Zhao ([25, Lemma 5.9]). By considering the case $\mathbb{k} = (k_1, \{1\}^{k_2-1})$ and $n = 1$ in the equality (42), we have

$$
\zeta_A(k_1, \{1\}^{k_2-1}) + (-1)^{k_2} \zeta_A^*(\{1\}^{k_2-1}, k_1) = 0
$$
since \( \zeta^*_A(\{1\}^k) = 0 \) for every positive integer \( k \). Therefore, the equality (42) is a generalization of Proposition 3.7 (30) since

\[
\zeta^*_A(\{1\}^{k_2-1}, k_1) = (-1)^{k_1 + k_2 - 1} \zeta^*_A(k_1, \{1\}^{k_2-1})
\]

by Proposition 3.7 (28). The equality (42) and its analogue for the usual multiple zeta values (the equality (9)) are consequences of the explicit formula of the antipode of the harmonic algebra or the Hopf algebra of quasi-symmetric functions ([5, Theorem 3.2] or [6, Theorem 3.1]). See also [30, Theorem 3], [7, Proposition 6], [10, Proposition 7,1], and [27, Proposition 3.7].

As an application of Remark 3.14 (37) and Corollary 3.16 (42), we give another proof of the following theorem which is a part of recent deep works by Kh. Hessami Pilehrood, T. Hessami Pilehrood, and Tauraso. The original proof is based on the identity [17, Theorem 2.2] which is different from our identities in Subsection 2.2.

**Theorem 3.18** ([17, Theorem 4.3]). Let \( k_1 \) and \( k_2 \) be positive integers satisfying the condition that \( k_1 + k_2 \) is even. Then we have

\[
\zeta^*_{A_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2} \left\{ 1 - (-1)^{k_2} \left( \frac{k_1 + k_2 + 1}{k_1 + 1} \right) \right\} \frac{B_{p-k_1-k_2-1}}{k_1 + k_2 + 1} \p,
\]

\[
\zeta^*_{A_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2} \left\{ 1 - (-1)^{k_2} \left( \frac{k_1 + k_2 + 1}{k_2 + 1} \right) \right\} \frac{B_{p-k_1-k_2-1}}{k_1 + k_2 + 1} \p.
\]

**Remark 3.19.** They also calculated \( \zeta^*_A(\{2\}^{k_1}, 3, \{2\}^{k_2}) \) and \( \zeta^*_A(\{2\}^{k_1}, 1, \{2\}^{k_2}) \) where \( k_1 \) and \( k_2 \) are positive integers and \( \bullet \in \{\emptyset, *\} \) ([17, Theorem 4.1 and Theorem 4.2]).

**Proof of Theorem 3.18.** Let \( k_1 \) and \( k_2 \) be positive integers such that \( k_1 + k_2 \) is even. Let \( w := k_1 + k_2 + 1 \). First, we show the star case. By Remark 3.14 (37), Proposition 3.7 (24), (26), and (27), we have

\[
\zeta^*_{A_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = -\zeta^*_{A_2}(k_1, k_2) - (\zeta^*_{A_2}(1, k_1, k_2) - \zeta^*_{A_2}(k_1 + 1, k_2)) \p
\]

\[
= -\frac{1}{2} \left\{ (-1)^{k_2} k_1 \left( \frac{w}{k_2} \right) - (-1)^{k_1} k_2 \left( \frac{w}{k_1} \right) - w + 1 \right\} \frac{B_{w-p}}{w} \p
\]

\[
- \left\{ -\frac{1}{2} \left( -1 \right)^{k_2} \left( \frac{w}{k_2} \right) + w \right\} \frac{B_{p-w}}{w} - (-1)^{k_1+1} \left( \frac{w}{k_1+1} \right) \frac{B_{p-w}}{w} \p
\]

\[
= \frac{1}{2} \left\{ 1 - (-1)^{k_2} \left( \frac{w}{k_2 + 1} \right) \right\} \frac{B_{p-w}}{w} \p.
\]

Let \( k = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) =: (l_1, \ldots, l_{w-2}) \). By Corollary 3.16 (42), we have

\[
\zeta_{A_2}(k) = \zeta^*_{A_2}(\bar{k}) + \sum_{j=1}^{w-3} (-1)^j \zeta_{A_2}(l_1, \ldots, l_j) \zeta^*_{A_2}(l_{w-2}, \ldots, l_{j+1}).
\]

We see that one of the following cases is satisfied for \( j = 1, \ldots, w-3 \):

(i) At least one of \( \zeta_{A_2}(l_1, \ldots, l_j) \) and \( \zeta^*_{A_2}(l_{w-2}, \ldots, l_{j+1}) \) is zero,
(ii) Both of $\zeta_{A_2}(l_1, \ldots, l_j)$ and $\zeta^*_{A_2}(l_{w-2}, \ldots, l_{j+1})$ belong to $pA_2$.
Therefore, the summation in the equality (45) vanishes and we have
$$\zeta_{A_2}([k]) = \zeta^*_{A_2}([k]).$$
This completes the proof.

3.3. Functional equations for the index $\{1\}^m$
In this subsection, we argue about functional equations of FMPs of the index $\{1\}^m$.

**Lemma 3.20.** Let $m$ be a positive integer. Then
\begin{align*}
\tilde{L}^*_{A,\{1\}^m}(t) &= L_{A,m}(1 - t), \\
L_{A,\{1\}^m}(t) &= (-1)^{m-1} L_{A,m}(1 - t).
\end{align*}

*Proof.* By Proposition 3.7 (23), the cases $k = \{1\}^m$ in theorem 1.3 (1) and (2) give the equalities (46) and (47), respectively.

By Lemma 3.20 and Proposition 3.11 (33), we can express every FMP of the index $\{1\}^m$ by a FP. Therefore, we can obtain functional equations of FMPs of the index $\{1\}^m$ from functional equations of FPs. For example, we get distribution properties for FMPs
of the index $\{1\}^m$ by the following result by Elbaz-Vincent and Gangl:

**Proposition 3.21** (Elbaz-Vincent and Gangl [2, Proposition 5.7 (2)]). Let $n$ be a non-zero integer and $m$ a positive integer. Let $\zeta_n$ be a primitive $|n|$-th root of unity. Then we have the following equality in $A_{Z[\zeta_n]}$ :
$$L_{A,m}(t^n) = n^{m-1} \sum_{k=0}^{\lfloor n/|n| \rfloor - 1} \frac{1 - t^{np}}{1 - (\zeta_n^k t)^p} L_{A,m}(\zeta_n^k t).$$

**Theorem 3.22** (Distribution properties for FMPs of the index $\{1\}^m$). Let $n$ be an non-zero integer and $m$ a positive integer. Let $\zeta_n$ be a primitive $|n|$-th root of unity. Then the following equalities hold in $A_{Z[\zeta_n]}$ :
\begin{align*}
L_{A,\{1\}^m}(1 - t^n) &= n^{m-1} \sum_{k=0}^{\lfloor n/|n| \rfloor - 1} \tilde{L}_{A,\{1\}^m} \left( \frac{1}{1 - \zeta_n^k t} \right), \\
L^*_{A,\{1\}^m} \left( \frac{1}{1 - t^n} \right) &= n^{m-1} \sum_{k=0}^{\lfloor n/|n| \rfloor - 1} L^*_{A,\{1\}^m} \left( \frac{1}{1 - \zeta_n^k t} \right), \\
\tilde{L}^*_{A,\{1\}^m} \left( 1 - t^n \right) &= n^{m-1} \sum_{k=0}^{\lfloor n/|n| \rfloor - 1} \frac{1 - t^{np}}{1 - (\zeta_n^k t)^p} \tilde{L}^*_{A,\{1\}^m} \left( 1 - \zeta_n^k t \right).
\end{align*}
Proof. These are obtained by Proposition 3.21. Note that \((1 - t^n)^p = 1 - t^{np}\) and \((1 - \zeta^k t)^p = 1 - (\zeta^k t)^p\) in \(A_{\mathbb{Z}[[t]]}\).

**Corollary 3.23.** Let \(m\) be a positive integer. Then the following equalities hold in \(A_{\mathbb{Z}[t]}:\)

\[
\bar{L}_{A,\{1\}}(t) = (-1)^{m-1}L_{A,\{1\}}(1 - t),
\]

\[
L^*_{A,\{1\}}(t) = (-1)^{m-1}L^*_{A,\{1\}}(1 - t).
\]  

_Proof._ Let \(n = -1\) in Theorem 3.22. Then we have the desired formulas by replacing \(1/(1 - t)\) with \(t\).

**Remark 3.24.** Lemma 3.20 (47) has been proved by Mattarei and Tauraso ([24, The proof of Theorem 2.3], [15, Lemma 3.2]) and Lemma 3.23 (53) has been proved by L. L. Zhao and Z. W. Sun ([29, Theorem 1.2]).

### 3.4. Relation between Ono-Yamamoto’s FMPs and our FMPs

Ono and Yamamoto gave another definition of finite multiple polylogarithms in the paper [18]. Their purpose is to establish the shuffle relation of FMPs ([18, Theorem 3.6]). In this subsection, we give the relation between Ono-Yamamoto’s FMPs and our FMPs.

**Definition 3.25.** Let \(k = (k_1, \ldots, k_m)\) be an index. Then Ono-Yamamoto’s finite multiple polylogarithm \(\ln_k(t) \in A_{\mathbb{Z}[t]}\) is defined by

\[
\ln_k(t) := \left( \sum_{0 < l_1, \ldots, l_m < p} \frac{t^{l_1 + \cdots + l_m}}{l_1^{k_1} (l_1 + l_2)^{k_2} \cdots (l_1 + \cdots + l_m)^{k_m}} \right) \bigg|_{p}
\]

where the summation \(\sum'\) runs over only fractions whose denominators are prime to \(p\).

By the substitution \(l_i \mapsto p - l_i\), we have

\[
\ln_k(r) = (-1)^{\text{wt}(k)} r^{\text{dep}(k)} \ln_k(r^{-1})
\]

for any non-zero rational number \(r\).

We prepare the following notations to discuss the relation between Ono-Yamamoto’s FMPs and our FMPs (cf. [18, Section 2]):

\[
[l] := \{1, \ldots, l\},
\]

\[
\Phi_{m,t} := \{\phi: [m] \to [l] : \text{surjective} \mid \phi(a) \neq \phi(a + 1) \text{ for all } a \in [m - 1]\},
\]

\[
m_\phi := l \text{ when } \phi \in \Phi_{m,t},
\]

\[
\Phi_m := \bigsqcup_{t=1}^{m} \Phi_{m,t}, \quad \delta_\phi(i) := \#\{a \in [i - 1] \mid \phi(a) > \phi(a + 1)\} \text{ for } \phi \in \Phi_m,
\]

\[
\beta: \Phi_m \to [m] \text{ is defined by } \beta(\phi) := \delta_\phi(m) + 1, \quad \Phi^i_m := \beta^{-1}(i),
\]

where \(i, l, m\) are positive integers.
Proposition 3.26. Let \( \mathbb{k} = (k_1, \ldots, k_m) \) be an index. Then we have

\[
\lim_k(t) = \sum_{i=1}^m t(i-1)p \sum_{\phi \in \Phi_m} \ell_{A, (\sum_{\phi(j)=m} k_j, \ldots, \sum_{\phi(j)=1} k_j)} \left( \left\{ 1 \right\}^{m_p \cdot \phi(m)}, t, \left\{ 1 \right\}^{\phi(m)-1} \right).
\] (56)

Proof. We omit the proof because it is completely the same as the proof of [18, Proposition 2.4]. \qed

By the following proposition, we see that Ono-Yamamoto's FMPs do not give FMZVs at \( t = 1 \) in general (cf. Remark 3.10).

Proposition 3.27. Let \( \mathbb{k} = (k_1, \ldots, k_m) \) be an index. Then we have

\[
\lim_k(1) = 0.
\] (57)

Proof. If \( m = 1 \), we have \( \lim_k(1) = \zeta_A(\mathbb{k}) = 0 \). We assume that \( m \) is greater than or equal to 2. Let \( l \) be one of 2, \ldots, \( m \) and \( S_l \) the \( l \)-th symmetric group. We define an equivalence relation on \( \Phi_{m,l} \) as follows: \( \phi \sim \phi' \) holds for \( \phi, \phi' \in \Phi_{m,l} \) if and only if there exists \( \sigma \in S_l \) such that \( \phi = \sigma \circ \phi' \) holds. We take and fix a system of representatives \( \{ \phi_{l,1}, \ldots, \phi_{l,i_l} \} \) of the quotient set \( \Phi_{m,l}/S_l \) where \( i_l \) is the cardinality of \( \Phi_{m,l}/S_l \). Then, by Proposition 3.26, we have

\[
\lim_k(1) = \sum_{l=2}^m \sum_{s=1}^{i_l} \zeta_A \left( \sigma \left( \sum_{\phi_{l,s}(j)=l} k_j, \ldots, \sum_{\phi_{l,s}(j)=1} k_j \right) \right).
\]

We see that this is zero by [6, Theorem 4.4]. \qed

4. Special values of finite multiple polylogarithms

We calculate some special values of F(S)MPs in \( A \) and \( A_2 \) by applying our main results.

Lemma 4.1 (Tauraso and J. Zhao [25]). Let \( m \) be an integer greater than 1. Let \( k_1 \) and \( k_2 \) be positive integers such that \( w := k_1 + k_2 \) is odd. Then we have the following equalities:

\[
\ell_{A,m}(-1) = \frac{1 - 2^{m-1} B_p - m}{2^{m-2}}
\] (58)

\[
\ell_{A,(k_1,k_2)}(-1) = \ell_{A,(k_1,k_2)}^\ast(-1) = \frac{2^{w-1} - 1}{2^{w-1}} B_p - \frac{m}{w}.
\] (59)

\[
\ell_{A^\ast,(k_1,k_2)}(-1) = \frac{1 - 2^{w-1}}{2^{w-1}} B_p - \frac{m}{w}.
\] (60)

If \( m \) is even, we have

\[
\ell_{A_2,m}(-1) = \frac{m(1 - 2^m)}{2^m} B_p - \frac{m+1}{m+1}.
\] (61)
Corollary 2.3].

First, we prove the star cases. We use the functional equation (1) for an index
Proof. Let \( t \).

Proposition 4.2. Let \( m \) be an integer greater than 1. Let \( k_1 \) and \( k_2 \) be integers such that \( w := k_1 + k_2 \) is odd. Then we have the following equalities:

\[
\mathcal{L}_{\mathcal{A}, \{1\}}^*(2) = \mathcal{L}_{\mathcal{A}, \{1\}}(2) = \frac{1 - 2^{m-1} B_{p-m}}{2^{m-2}} \frac{B_{p-m}}{m},
\]

\[
\mathcal{L}_{\mathcal{A}, \{1\}}^*(1/2) = \mathcal{L}_{\mathcal{A}, \{1\}}(1/2) = \frac{2^{m-1} - 1}{2m-1} \frac{B_{p-m}}{m},
\]

\[
\mathcal{L}_{\mathcal{A}, \{1\}}^*(1/2) = \frac{1}{2} \left( \frac{2^{w-1} - 1}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right) \frac{B_{p-w}}{w},
\]

\[
\mathcal{L}_{\mathcal{A}, \{1\}}^*(1/2) = \frac{1}{2} \left( \frac{2^{w-1} - 1}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right) \frac{B_{p-w}}{w}.
\]

Proof. First, we prove the star cases. We use the functional equation (1) for an index \( k' \):

Consider the case \( t = 2 \) and \( k = m \) of the equality (69) (or the equality (46)). Then we obtain the star case of the equality (63) by Lemma 4.1 (58). Considering the case \( k = (k_1, k_2) \) of the equality (69), we obtain the equality (66) by Lemma 4.1 (59) and Proposition 3.7 (24). The star case of the equality (64) and the equality (68) are obtained by Proposition 3.11.

Next, we prove the non-star cases by Corollary 3.16 (40) for \( \mathcal{A} \). By considering the case \( t = 2 \) and \( k = \{1\}^m \) (i.e. the case \( t = 2 \) of the equality (47)), we have the non-star case of the equality (63). We consider the case \( k = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) \) satisfying the condition that \( k_1 + k_2 \) is odd. The summation for \( j = 1, \ldots, w-2 \) of the right hand side of (40) vanishes since \( \zeta_\mathcal{A}(\{1\}^{w-1-j}) = 0 \). We suppose that \( j \) is an element of \( \{0, \ldots, k_1 - 1\} \). If \( j \) is odd, we have \( \zeta_\mathcal{A}(\{1\}^{k_2-1}, 2, \{1\}^{k_1-j-1}) = 0 \) by Theorem 3.18 and if \( j \) is even, we have \( \mathcal{L}_{\mathcal{A}, \{1\}}(2) = 0 \) by the equality (63). Hence, we see that the summation in the right hand side of (40) vanishes and we have the equality (65). The non-star case of the equality (64) and the equality (67) are obtained by Proposition 3.11.

\[
\mathcal{L}_{\mathcal{A}, \{1\}}^*(1/2) = \mathcal{L}_{\mathcal{A}, \{1\}}(1/2) = \frac{1}{2} \left( \frac{2^{w-1} - 1}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right) \frac{B_{p-w}}{w}.
\]
Remark 4.3. Z. W. Sun proved that $\mathcal{L}_{A,\{1\}}^*(1/2) = 0$ (see [22, Theorem 1.1]). The proof is based on some technical calculations. The case $(k_1, k_2) = (1, 2)$ or $(k_1, k_2) = (2, 1)$ of Proposition 4.2 (66) and (65) have already been obtained by Meštrović [14, Theorem 1.1, Corollary 1.2] and by Tauraso and J. Zhao [25, Proposition 7.1].

Definition 4.4. Let $n$ be a positive integer and $a$ a non-zero rational number. We define the element $q_p(a)$ of $A_n$ to be $(q_p(a) \mod p^n)_p$ where $q_p(a)$ is the Fermat quotient, that is,

$$q_p(a) = \frac{a^{p-1} - 1}{p}$$

for a prime number $p$.

By Fermat’s little theorem, $q_p(a)$ is well-defined as an element of $A_n$. Under the hypothesis that $abc$-conjecture is true, we see that $q_p(a)$ is non-zero. See [19].

Theorem 4.5. Let $m$ be a positive even number. Then we have the following equalities in $A_2$:

$$\mathcal{L}_{A_2,\{1\}}(2) = -\mathcal{L}_{A_2,\{1\}}^*(2) = \left(\frac{m+1}{2^m} - m - 2\right) \frac{B_{p-m-1}}{m+1} p, \quad (70)$$

$$\mathcal{L}_{A_2,\{1\}}^*(1/2) = -\mathcal{L}_{A_2,\{1\}}^*(1/2) = \frac{1 - 2^{m+1} B_{p-m-1}}{2^{m+1} m+1} p. \quad (71)$$

Proof. First, we prove the star cases. By the functional equation Remark 3.14 (36), we have

$$\tilde{\mathcal{L}}_{A_2,\{1\}}^*(2) - \zeta_{A_2}(\{1\})^m = \mathcal{L}_{A_2,m}(-1) + (\tilde{\mathcal{L}}_{A_2,\{1\},m}(-1) - \mathcal{L}_{A_2,m+1}(-1))p.$$

Therefore, by combining Proposition 3.7 (23), Lemma 4.1 (58), (59) and (61), we have

$$\tilde{\mathcal{L}}_{A_2,\{1\}}^*(2) - \tilde{\mathcal{L}}_{A_2,\{1\}}(2) = \frac{m(2m - 1)}{2^m} \frac{B_{p-m-1}}{m+1} p + \left(1 - \frac{2^m B_{p-m-1}}{2^{m+1}} - \frac{1 - 2^m B_{p-m-1}}{m+1}\right) p,$$

or

$$\tilde{\mathcal{L}}_{A_2,\{1\}}^*(2) = \left(m + 2 - \frac{m+1}{2^m}\right) \frac{B_{p-m-1}}{m+1} p. \quad (72)$$

By the equality (31), we have

$$\mathcal{L}_{A_2,\{1\}}(2) = \frac{1}{2p} \left(\tilde{\mathcal{L}}_{A_2,\{1\}}^*(2) + p \sum_{i=1}^m \tilde{\mathcal{L}}_{A_2,\{1\},i-2,\{1\}^{m-i}}(2)\right).$$

Hence, by combining the equality (72) and Proposition 4.2 (66), we have

$$\mathcal{L}_{A_2,\{1\}}^*(1/2) = \frac{1}{2} \left(\frac{m + 2 - \frac{m+1}{2^m}}{2^m} \frac{B_{p-m-1}}{m+1} p + \sum_{i=1}^m \left(1 - \frac{2^m}{2^{m+1}} - (-1)^i \frac{m+1}{i}\right) \frac{B_{p-m-1}}{m+1}\right)$$

$$= \frac{2^{m+1} - 1 B_{p-m-1}}{m+1} p.$$
since $m$ is even and $\sum_{i=1}^{m}(-1)^i(m+1) = 0$. Note that the equality $2^p = 2(1 + q_p(2)p)$ holds in $A_2$ and $E_{A_2,\{1\}^m}(1/2) = 0$.

Next, we prove the non-star cases. By Corollary 3.16 (40) for $A_2$, we have

$$L_{A_2,\{1\}^m}(2) = -L_{A_2,\{1\}^m}(2) + \sum_{j=1}^{m-1}(-1)^{j-1}L_{A_2,\{1\}^j}(2)\zeta_{A_2}^*(\{1\}^{m-j}). \quad (73)$$

Since $\zeta_{A_2}^*(\{1\}^{m-j})$ is contained in $pA_2$, we have

$$L_{A_2,\{1\}^j}(2)\zeta_{A_2}^*(\{1\}^{m-j}) = (\text{a certain rational number}) \times B_{p-m+j-1}B_{p-j}p$$

for any $j = 1, \ldots, m - 1$ by Proposition 3.7 (23) and Proposition 4.2 (63). If $j$ is odd, we have $B_{p-m+j-1} = 0$ and if $j$ is even, we have $B_{p-j} = 0$ because $B_{2n+1} = 0$ for any positive integer $n$. Therefore, the summation in the right hand side of (73) vanishes and we have

$$L_{A_2,\{1\}^m}(2) = -\tilde{L}_{A_2,\{1\}^m}(2) = \left(\frac{m+1}{2^m} - m - 2\right) \frac{B_{p-m-1}}{m+1}p. \quad (74)$$

by Proposition 4.2 (63) and (65), Proposition 3.7 (23), and Lemma 4.1 (61). By the equality (31), the equality (74), and Proposition 4.2 (65), we also have

$$\tilde{L}_{A_2,\{1\}^m}(1/2) = \frac{1}{2p}L_{A_2,\{1\}^m}(2) + p\sum_{i=1}^{m}L_{A_2,\{1\}^{i-1,2,\{1\}^{m-i}}}(2)$$

$$= \frac{1}{2}\left\{\left(\frac{m+1}{2^m} - m - 2\right) B_{p-m-1}p + \sum_{i=1}^{m} \left(\frac{2^m - 1}{2^m} - (-1)^i\binom{m+1}{i}\right) B_{p-m-1}p\right\}$$

$$= \frac{1 - 2^{m+1}B_{p-m-1}}{m+1}p. \quad \square$$

**Remark 4.6.** The cases $m = 2$ of Theorem 4.5 have already been given by Z. W. Sun and L. L. Zhao [23], Meštrović [14], and Tauraso and J. Zhao [25]. Indeed, the non-star case of the equality (70) is [25, Proposition 7.1 (78)] and the star case of the equality (70) is [14, Theorem 1.1 (1)] or [25, Proposition 7.1(77)]. The star case of Theorem 4.5 (71) was conjectured by Z. W. Sun [28, Conjecture 1.1] and proved by Z. W. Sun and L. L. Zhao [23]. Meštrović gave another proof of Sun’s conjecture in [14] and our proof of the equality (71) is similar to his proof.

**Remark 4.7.** We can calculate many other special values of FMPs including the following two equalities by combining our main results and Tauraso-Zhao’s results:

$$L_{A_2,\{1\}^{w'}}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-2}, 1/2, 2, \{1\}^{k_3-1}) = \frac{1 - 2^{w' - 1}}{2^{w'}} \left\{\binom{w'}{k_1} - \binom{w'}{k_3}\right\} B_{p-w'}, \quad (75)$$

$$L_{A_2,\{1\}^{w-1}}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = -\frac{1}{2} \left\{1 + \frac{2^{w-1} - 1}{2^{w-1}}\left(\binom{w}{k_2}\right)\right\} B_{p-w}p. \quad (76)$$

Here, $w' = k_1 + k_2 + k_3$, $w = k_1 + k_2 + 1$, and we assume that $k_1$ is even, $k_2$ is odd, $k_3$ is even, and $k_2 > 1$ in (75) and $k_1$ and $k_2$ are even in (76), respectively. Other values can be found on the web page http://arxiv.org/pdf/1509.07653.pdf.
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